Lecture 1

Continuum Mechanics Background

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Kinematics : deformations and motions

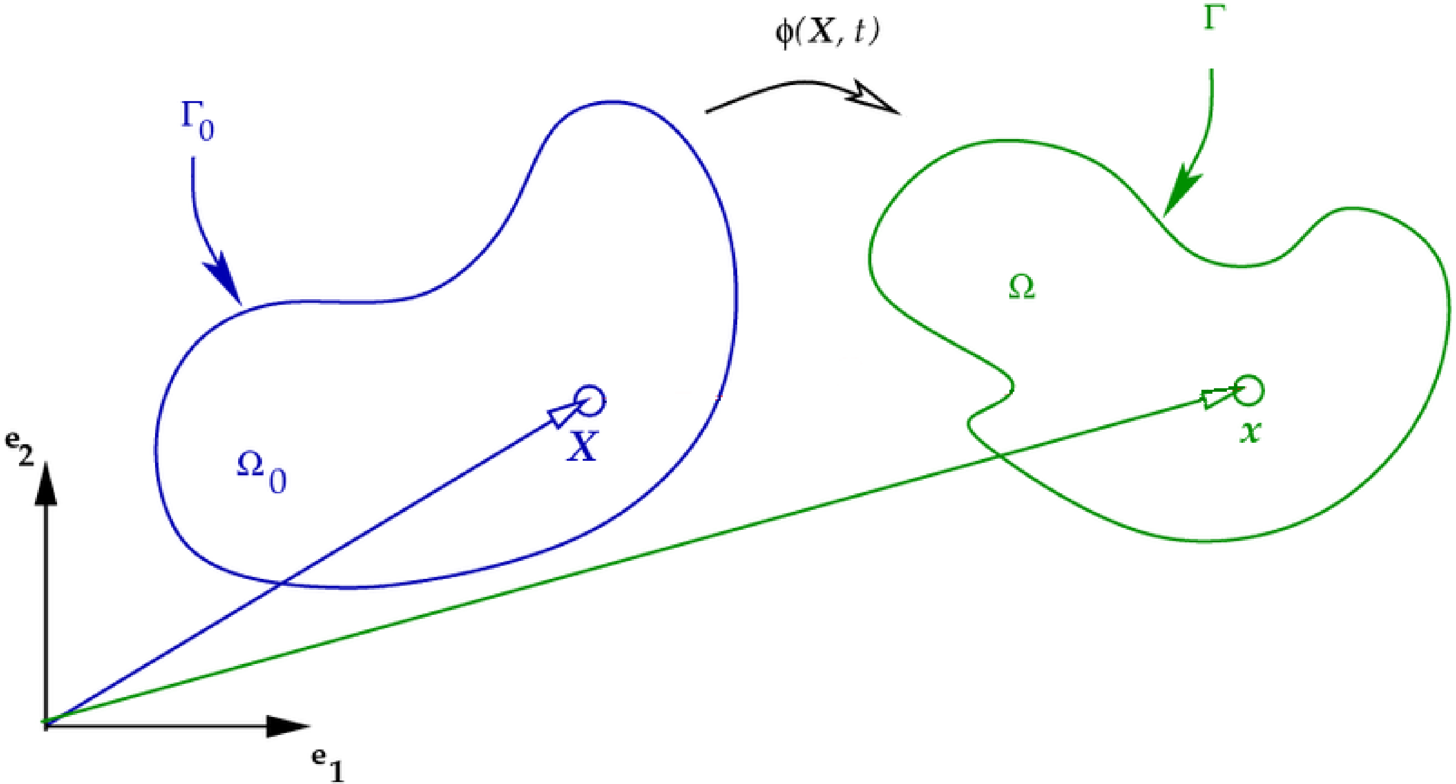
Configurations of a continuos medium

Ω_0 The reference configuration

Ω_t: deformed configuration

X: position vector of a particle at t=0

x: Curent position



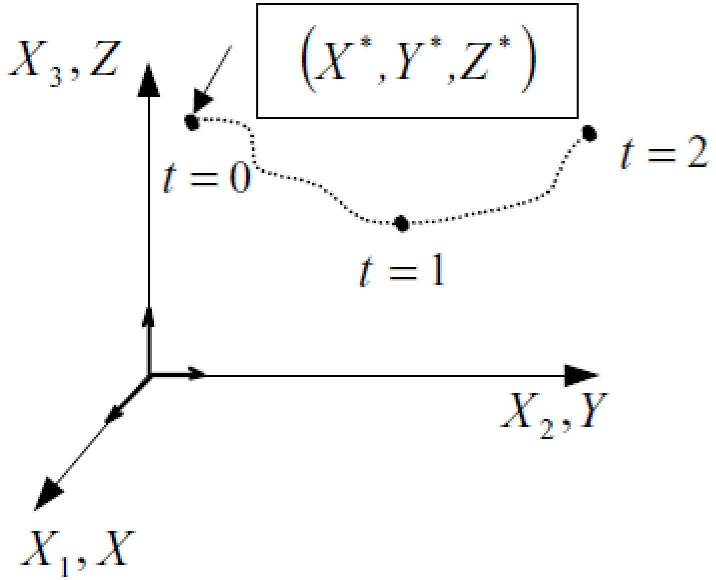
Admissible motions

- Consistency condition : $\Phi(0, X) = X$ for all $X \in \Omega_0$;
- Smoothness : $\Phi \in C^k$ with k as large as needed;
- Biunivocity : Φ is one-to-one (two particles cannot occupy simultaneously the same spot in space).
- $F(t, X) = \nabla\Phi(t, X) = \left[\frac{\partial \Phi_i}{\partial X_j}(t, X)\right]$ satisfies $\det F(t, X) > 0$.

(see later that this is the condition to have a positive density)

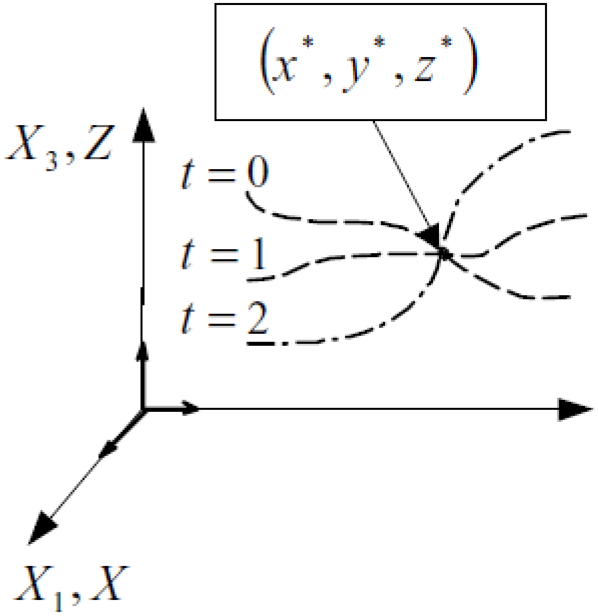
Material or Lagrangian description of the motion

- The physical properties are described in terms of particles positions X and of time.
- It focusses on what happens at a fixed particle, labeled by its material coordinates, as time progresses.
- Generally used in solid mechanics.



Spatial or Eulerian Description

- The physical properties are described in terms of the spatial coordinates and time.
- It focusses on what happens at a fixed point in space (labeled by its spatial coordinates) as time progresses.
- Generally used in fluid mechanics.



Velocity and acceleration

- Lagrangian velocity

$$V(t,X) = \frac{\partial \Phi}{\partial t}(t,X)$$

- Lagrangian acceleration

$$A(t,X) = \frac{\partial V}{\partial t}(t,X) = \frac{\partial^2 \Phi}{\partial t^2}(t,X)$$

- Eulerian velocity

$$v(t,x) = V\big(t,\Phi^{-1}(t,x)\big)$$

$$V(t,X) = v\big(t,\Phi(t,X)\big).$$

- Eulerian acceleration

$$a(t,x) = A\big(t,\Phi^{-1}(t,x)\big)$$

$$A(t,X) = a\big(t,\Phi(t,X)\big)$$

Eulerian acceleration and Eulerian velocity

Proposition 1. For every $i \in \{1,2,3\}$, $t \geq 0$ and $x \in \Omega$ we have

$$a_i(t,x) = \frac{\partial v_i}{\partial t}(t,x) + \sum_{k=1}^{3} v_k(t,x)\frac{\partial v_i}{\partial x_k}(t,x)$$

or

$$a(t,x) = \frac{\partial v}{\partial t}(t,x) + \big((v \cdot \nabla)v\big)(t,x),$$

where we have denoted

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix}$$

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Proof Proposition 1

Recall that $V(t,X) = v\big(t,\Phi(t,X)\big)$. Thus, by the chain rule

$$A_i(t,X) = \frac{\partial V_i}{\partial t}(t,X) = \frac{\partial v_i}{\partial t}\big(t,\Phi(t,X)\big) + \sum_{k=1}^{3} \frac{\partial v_i}{\partial x_k}\big(t,\Phi(t,X)\big)V_k(t,X).$$

The conclusion follows using the fact that

$$a(t,x) = A\big(t,\Phi^{-1}(t,x)\big).$$

Isochoric motions and incompressible media

Definition 1. A motion Φ is said **isochoric** if for every open set $P_0 \subset \Omega_0$ and every $t \geq 0$ we have

$$\text{vol}\,(P_t) = \text{vol}\,(P_0),$$

where we used the notation $P_t = \Phi(t, P_0)$. A continuous medium is said **incompressible** if it undergoes only isochoric motions.

Proposition 2. A motion Φ is **isochoric** if and only if

$$\text{div}\, v(t,x) = 0$$

for every $t \geq 0$ and $x \in \Omega_t$.

Proof of Proposition 2 (1)

Lemma 1. Recall that $F(t,X) = (\nabla_X \Phi)(\mathrm{t,X})$ and let $L(t,x) = \nabla_x v(t,x)$. Then for every $t \geq 0$ and $X \in \Omega_0$ we have:

$$\frac{\partial F}{\partial t}(t,X) = L\big(t,\Phi(t,X)\big)F(t,X).$$

Proof.

$$\frac{\partial F}{\partial t}(t,X) = \nabla_X\left(\frac{\partial \Phi}{\partial t}(t,X)\right) = \nabla_X(v\,(t,\Phi(t,X)) = (\nabla_x v)\big(t,\Phi(t,X)\big)\nabla_X\Phi(t,X)$$

$$= \quad L\big(t,\Phi(t,X)\big)F(t,X).$$

Proof of Proposition 2 (2)

We first note that Φ is isochoric iff for every $t \geq 0$ and $P_0 \subset \Omega_0$ we have

$$\int_{P_t} dx = \int_{P_0} dX.$$

By the change of variables formula, this is equivalent to

$$\int_{P_0} \det F(t,X)\, dX = \int_{P_0} dX,$$

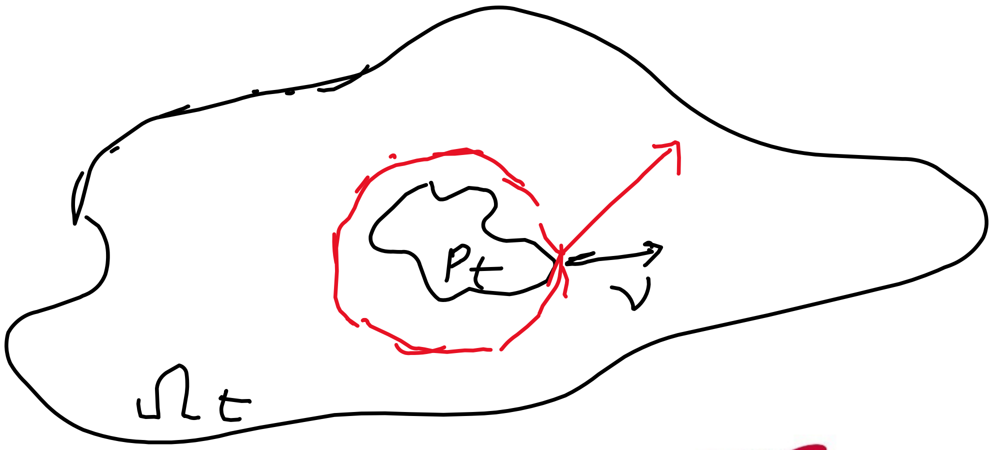
thus equivalent to $\det F(t,X) \equiv 1.$ Thus Φ is isochoric iff $\frac{\partial}{\partial t}(\det F(t,X)) \equiv 0,$ which is equivalent to $\operatorname{tr}\left[\frac{\partial F}{\partial t}(t,X)F^{-1}(t,X)\right] \equiv 0$.

We conclude using Lemma 1.

Dynamics :
the principles of continuum mechanics

Balance of mass and momentum (integral forms)

- Balance of mass: $\frac{d}{dt}(m(P_t)) = \frac{d}{dt}\int_{P_t} \rho(t,x)\,dx = 0.$
- Balance of linear momentum: $\frac{d}{dt}\int_{P_t} \rho v\,dx = \int_{P_t} \rho b\,dx + \int_{\partial P_t} s\,(t,x,\nu)da.$
- Balance of angular momentum: $\frac{d}{dt}\int_{P_t} x \times \rho v\,dx = \int_{P_t} \rho x \times b\,dx + \int_{\partial P_t} x \times s\,(t,x,\nu)da.$



Cauchy's postulate

Local form of the balance of mass

$$0 = \frac{d}{dt}\int_{P_t} \rho(t,x)\,dx = \frac{d}{dt}\int_{P_0} \rho(t,\Phi(t,X))\det F(t,X)\,dX =$$

$$= \int_{P_o} \left(\frac{\partial \rho}{\partial t}(t,\Phi(t,X)) + \nabla_x \rho\big(t,\Phi(t,X)\big) v(t,\Phi(t,X))\right.$$

$$\left. + \rho\big(t,\Phi(t,X)\big)\mathrm{tr}\left(\frac{\partial F}{\partial t}(t,X) F^{-1}(t,X)\right)\right)\det F(t,X)dX$$

$$= \int_{P_t} \left(\frac{\partial \rho}{\partial t}(t,x) + \mathrm{div}\,(\rho v)(t,x) \right) dx$$

We have thus proved the <u>continuity equation</u>, which holds for $t \geq 0$ and $x \in \Omega_t$:

$$\partial \rho$$

Local form of the balance of linear momentum

Theorem 1 (Cauchy, 1821). For every $t \geq 0$ there exists a tensor field (called **stress tensor**)

$$\sigma(t,\cdot): \Omega_t \to L(\mathbb{R}^3)$$

such that for every $t \geq 0$ and $x \in P_t$ we have

$$s\big(t, x, \nu(x)\big) = \sigma(t,x)\nu(t,x).$$

Corollary 1 (local for of the balance of the linear momentum).

For every $t \geq 0$ and $x \in \Omega_t$ we have:

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = \operatorname{div}\sigma + \rho b.$$

Proof of Corollary 1 (1)

$$\frac{d}{dt}\int_{\mathrm{P}_t} \rho(t,x)v(t,x)dx = \frac{d}{dt}\int_{\mathrm{P}_0} \rho\big(t,\Phi(t,X)\big)\,)\,\det F(t,X)\, v(t,\Phi(t,X)\; dX$$

$$= \int_{\mathrm{P}_0} \left[\frac{\partial\rho}{\partial t}\big(t,\Phi(t,X)\big)v\big(t,\Phi(t,X)\big) + \nabla\rho\big(t,\Phi(t,X)\big)\cdot v\big(t,\Phi(t,X)\big)\right] \det F(t,X)\;\; v(t,\Phi(t,X))\; dX$$

$$+\int_{\mathrm{P}_0} \rho\big(t,\Phi(t,X)\big)\;\det F(t,X)\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right](t,\Phi(t,X))\; dX$$

$$+\int_{\mathrm{P}_0} \rho\big(t,\Phi(t,X)\big)\mathrm{tr}\left(\frac{\partial F}{\partial t}(t,X)F^{-1}(t,X)\right)\det F(t,X)\;\; v\big(t,\Phi(t,X)\big)\; dX$$

$$= \int_{\mathrm{P}_0} \rho\big(t,\Phi(t,X)\big)\;\det F(t,X)\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right]\big(t,\Phi(t,X)\big)dX = \int_{\mathrm{P}_t}\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right](t,x)dx$$

Proof of Corollary 1 (2)

We next remark that $\int_{\partial P_t} \sigma \nu \, \mathrm{da} = \int_{P_t} \mathrm{div}\, \sigma \, \mathrm{d}x$ so that

$$\int_{P_t} \left(\frac{\partial \rho}{\partial t}(t,x) + \mathrm{div}\,(\rho v)(t,x) \right) dx = \int_{P_t} (\rho b \; + div\ \sigma)\, \mathrm{d}x.$$

Since the above formula folds for every $t \geq 0$ and $P_0 \subset \Omega_0$ it follows that indeed

$$\rho \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = \mathrm{div}\, \sigma + \rho b.$$

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Local form of the balance of angular momentum

Theorem 1 (Cauchy, 1821). For every $t \geq 0$ and $x \in \Omega_t$ we have

$$\sigma(t,x) = \sigma^*(t,x),$$

i.e., the stress tensor is symmetric.

The general PDEs of continuum mechanics

- Balance of mass: $\frac{\partial \rho}{\partial t}(t,x) + \mathrm{div}\,(\rho v)(t,x) = 0$

- Balance of linear momentum: $\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = \mathrm{div}\,\sigma + \rho b.$

- Balance of angular momentum: $\sigma(t,x) = \sigma^{\,*}(t,x).$

Constitutive assumptions for ideal and viscous fluids.
Navier-Stokes Equations

Ideal incompressible fluids (I)

- For incompressible media we have from Proposition 2 an extra equation:

$$\operatorname{div} v(t,x) = 0.$$

- If the fluid is homogeneous (that is $\rho(0,x) = \rho_0 > 0$) then $\rho(t,x) = \rho_{0.}$

- In ideal fluids the stress tensor is supposed to be a spherical one, i.e.,

$$\sigma(t,x) = -p(t,x)\,\boldsymbol{I}_3\,.$$

Ideal incompressible fluids (II)

An ideal incompressible fluid is thus described by the Euler equations (1757):

$$\begin{cases} \rho\left(\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v\right) = -\nabla p + \rho b & (t \geq\ 0,\ \mathrm{x} \in \Omega_t), \\ \operatorname{div} v(t,x) = 0 & (t \geq 0, x \in \Omega_t), \end{cases}$$

with the boundary conditions

$v \cdot \nu = 0$ on $\partial\Omega_t$.

The analysis of the Euler equations is still an active research field.

Potential flows (I)

Definition.
An ideal fluid flow is said *potential* if there exists $\varphi(t, \mathbf{x})$ such that $v = \nabla\varphi$.

Proposition. In a simply connected domain, $v = \nabla\varphi$ satisfies the Euler equations, with $b = 0$, for some p iff $\Delta\varphi = 0$. In this case

$$p = \frac{\partial\varphi}{\partial t} + c(t).$$

Proposition (Bernoulli). In a potential flow we have

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + p = c(t).$$

Potential flows (II)

Potential flows (III)

Proposition.
In a simply connected domain a flow is potential iff $v(0,\cdot)=\nabla\varphi_0$.

Viscous incompressible fluids (I)

The stress tensor σ is related to the velocity gradient and to the pressure by

$$\sigma = -p\, \boldsymbol{I}_3 + \mu\, (\nabla v + (\nabla v)^*).$$

A viscous incompressible fluid is thus described by the Navier-Stokes equations,1822 (Navier) to 1842–1850 (Stokes):

with the boundary conditions

$v = 0$ on $\partial\Omega_t$.

The analysis of the Navier-Stokes Equations is still an active research field. (millennium problem)

Dimensionless form of the Navier-Stokes equations and Stokes system

The basic initial and boundary value problem for the Navier-Stokes equations

$$\left\{\begin{array}{ll} \rho\left(\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v\right)-\nu\Delta v+\nabla p=f, & (t\geqslant 0, x\in\Omega), \\ \operatorname{div} v(t,x)=0 & (t\geqslant 0,\ x\in\Omega), \\ v(t,x)=0 & (t\geqslant 0,\ x\in\partial\Omega), \\ v(0,x)=v_0(x) & (x\in\Omega). \end{array}\right. \qquad (NSE)$$

Dimensionless variables

$$t^*=\omega t,\ x^*=L^{-1}x, v^*=V^{-1}v,\ p^*=\left(\rho V^2\right)^{-1}p,\ \nabla^*=L\nabla,$$

where L is a characteristic length, ω is a typical frequency and V is a characteristic speed.

The Navier-Stokes equations in dimensionless variables

$$\begin{cases} \sigma \dfrac{\partial v^*}{\partial t^*} + (v^* \cdot \nabla^*) v^* - \mathrm{Re}^{-1} \Delta v^* + \nabla p^* = f^*, & (t^* \geqslant 0, x^* \in \Omega^*), \\ \operatorname{div} v^* = 0 & (t^* \geqslant 0,\ x^* \in \Omega^*), \\ v^* = 0 & (t^* \geqslant 0,\ x^* \in \partial\Omega), \\ v^*(0, x^*) = v_0^*(x^*) & (x \in \Omega), \end{cases}$$

where $\sigma = \dfrac{\omega L}{\nu}$ is the frequency parameter characteristic length, $\mathrm{Re} = \dfrac{VL}{\nu}$ is the Reynolds number.

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The Stokes system

If σ is of order unity and Re $<<1$ the (NSE) becomes

$$\left\{\begin{array}{ll} -\nu\Delta v+\nabla p=f, & (x\in\Omega), \\ \operatorname{div} v=0 & (x\in\Omega), \\ v=0 & (x\in\partial\Omega). \end{array}\right. \qquad (Stokes)$$

In the above system f is given and the unknown are v and p.

Function spaces for the Stokes system.

The Helmholtz-Leray projector

Basic function spaces for the Stokes system

Let $n \in \{2,3\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$, with (at least) Lipschitz boundary. If there is no risk of confusion, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in $\left[L^2(\Omega)\right]^n$.

The following function spaces are of particular importance in mathematical fluid mechanics:

$$L^2_\sigma(\Omega) = \overline{C_0^\infty\underset{,\sigma}{(\Omega)}}^{\|\cdot\|_{[L^2(\Omega)]^n}}, \qquad C^\infty_{0,\sigma}(\Omega) = \{f \in [C_0^\infty(\Omega)]^n \ \mid \ \operatorname{div} f = 0\},$$

$$G(\Omega) = \left\{f \in \left[L^2(\Omega)\right]^n \ \mid \ \exists p \in \left[L^2(\Omega)\right]^n \text{s.t. } f = \nabla p\right\}.$$

$$\mathcal{V}(\Omega) = \left\{f \in \left[\mathcal{H}^1_0(\Omega)\right]^n \ \mid \ \operatorname{div} f = 0\right\}.$$

Free divergence (solenoidal) and irrotational vector fields (De Rham)

$H^1_0 \subset L^2 \subset H^{-1}$

Lemma 1. Let $f \in \left[\underset{\|}{\mathcal{H}^{-1}}(\Omega)\right]^n$ be such that

$$\langle f, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0 \qquad (\underset{v}{\cancel{f}} \in C^\infty_{0,\sigma}(\Omega)).$$

Then there exists a unique $p \in L^2_{\cancel{\mathrm{loc}}}(\Omega)$ such that $\nabla p = f$ in $\mathcal{D}'(\Omega)$ and

$$\int_\Omega p \,\mathrm{d}x = 0.$$

Moreover, there exist positive constants $C_1,\ C_2$, depending only on Ω, such that

$$\|p\|_{L^2(\Omega)} \leqslant C_1\|f\|_{[\mathcal{H}^{-1}(\Omega)]^n} \leqslant C_1C_2\|p\|_{L^2(\Omega)} \qquad \left(f \in \left[\mathcal{H}^{-1}(\Omega)\right]^n\right).$$

Proof. See, for instance, [Sohr, 2012], Section II.2.2.

$$\operatorname{div} v = 0 \Rightarrow \exists w \text{ s.t. } v = \operatorname{rot} w$$

$$\langle f, v \rangle = 0 \quad \forall v \in C^\infty_{0,\sigma}(\Omega) \Leftrightarrow$$

$$\Leftrightarrow \int_\Omega f \operatorname{rot} w = 0 \quad \forall w \in \mathcal{D}(\Omega)$$

$$\Rightarrow \int_\Omega (\operatorname{rot} f) \cdot w = 0 \quad \forall w \in \mathcal{D}(\Omega)$$

$$\Rightarrow \operatorname{rot} f = 0 \Rightarrow \exists p \text{ s.t. } f = \nabla p.$$

A density result

Lemma 2. $\mathcal{V}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{[\mathcal{H}^1_0(\Omega)]^n}}$.

Proof. Obviously $\overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{[\mathcal{H}^1_0(\Omega)]^n}} \subset \mathcal{V}(\Omega)$. To prove the opposite inclusion, let $f \in \underset{\mathcal{V}(\Omega)}{[\cancel{\mathcal{H}^{-1}(\Omega)}]^n}$ be such that

$$\langle f, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0 \qquad (\overset{v}{\cancel{f}} \in C^\infty_{0,\sigma}(\Omega)).$$

Using Lemma 1 it follows that there exists a unique $p \in L^2(\Omega)$ such that $\nabla p = f$ in $\mathcal{D}'(\Omega)$. Consequently

$$0 = \langle f, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = -\langle p, \operatorname{div} v\rangle \qquad (\overset{v}{\cancel{f}} \in C^\infty_{0,\cancel{\sigma}}(\Omega)).$$

The above formula holds, by density, for every $v \in \left[\mathcal{H}^1_0(\Omega)\right]^n$. Thus

$$\langle f, v\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0 \qquad (\overset{v}{\cancel{f}} \in \mathcal{V}(\Omega)). \Longrightarrow f = \{0\}$$

The Helmholtz-Leray decomposition

Lemma 3. $G(\Omega) = \left[L^2_\sigma(\Omega)\right]^\perp$.

Proof. The inclusion $\left[L^2_\sigma(\Omega)\right]^\perp \subset G(\Omega)$ follows from Lemma 1.
To prove that $G(\Omega) \subset \left[L^2_\sigma(\Omega)\right]^\perp$, let $f = \nabla p$, with $p \in L^2_{\mathrm{loc}}(\Omega)$. Then

$$\langle f, v\rangle = \langle \nabla p, v\rangle = -\langle p, \operatorname{div} v\rangle = 0 \qquad (v \in C^\infty_{\sigma,0}(\Omega)),$$

which ends the proof.

Corollary 1. Each $f \in \left[L^2(\Omega)\right]^n$ has a unique decomposition

$$f = f_0 + \nabla p,$$

with $f_0 \in L^2_\sigma(\Omega)$, $\nabla p \in G(\Omega)$, $\langle f_0, \nabla p\rangle = 0$ and

$$\|f\|^2 = \|f_0\|^2 + \|\nabla p\|^2.$$

The Helmholtz-Leray projector and a last lemma

Corollary 2. The operator $P : [L^2(\Omega)]^n \to L^2_\sigma(\Omega)$, defined by $Pf := f_0$ is linear and bounded linear with

$$\|Pf\| \leqslant \|f\| \qquad (f \in [L^2(\Omega)]^n).$$

Moreover, P has the following properties:

$$P(\nabla p) = 0, \quad (I - P)f = \nabla p, \quad P^2 f = Pf,$$

$$(I-P)^2 f = (I-P)f, \quad \langle Pf, g\rangle = \langle f, Pg\rangle, \quad \|f\|^2 = \|Pf\|^2 + \|(I-P)f\|^2,$$

for every $f,\ g \in [L^2(\Omega)]^n$.

$$L^2_\sigma(\Omega) = \{\varphi \in L^2(\Omega) \mid \int_\Omega \varphi = 0\}$$

Lemma 4. $L^2_\sigma(\Omega) = \{f \in [L^2(\Omega)]^n \mid \operatorname{div} f = 0, f \cdot \nu_{|\partial\Omega} = 0\}$.

Proof. See [Sohr, Section II.2.5].

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad f_k \in L^2(\Omega). \qquad \nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$\sum_{k=1}^{3} \frac{\partial f_k}{\partial x_k} \in L^2(\Omega) \not\Rightarrow$$

$$\xcancel{\frac{\partial f_j}{\partial x_j} \notin} \qquad \frac{\partial f_j}{\partial x_j} \in L^2(\Omega)$$

$$f \notin \left[H^1(\Omega)\right]^3$$

Normal trace for $f \in L^2(\Omega)$ with $\operatorname{div} f = 0$. $\quad f\cdot\nu \in H^{-\frac{1}{2}}(\partial\Omega)$ | $f\cdot\tau = ?$

$$\int_\Omega (\operatorname{div} f)\varphi \, dx = 0 \text{ for all } \varphi \in H^1(\Omega)$$

$$\Updownarrow$$

$$\cancel{\int_\Omega \operatorname{div}}(\varphi f) - \int_\Omega f\cdot\nabla\varphi = 0$$

$$\boxed{\int_{\partial\Omega} (f\cdot\nu)\varphi.} = \boxed{\int_\Omega f\cdot\nabla\varphi.} \quad \forall \varphi \in H^1(\Omega).$$

Linear bounded form on $H^1(\Omega)$

$\downarrow$

$\underline{H^{1/2}(\partial\Omega)}$

Analysis of the stationary Stokes system in bounded domains

The basic boundary value problem

We want to determine $v : \overline{\Omega} \to \mathbb{R}^n$ and $p : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} -\mu\Delta v + \nabla p &= f && (x \in \Omega), \\ \operatorname{div} v &= 0 && (x \in \Omega), \\ v &= 0 && (x \in \partial\Omega) \end{aligned}$$

Theorem 1. For every $f \in [L^2(\Omega)]^n$ there exists a unique solution $(v, p) \in \mathcal{V}(\Omega) \times L^2_0(\Omega)$. Moreover, there exists $C := C(\Omega) > 0$ such that

$$\|v\|_{\mathcal{V}(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\|f\|_{[L^2(\Omega)]^n} \qquad \left(f \in \left[L^2(\Omega)\right]^n\right).$$

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Proof. According to Lemma 3, (v, p) sollution iff

$$\langle \mu \Delta v + f, \psi \rangle = 0 \qquad \forall\, \psi \in \mathcal{V}(\Omega)\,.$$

This is in turn equivalent to

$$\mu \int_\Omega \nabla v \cdot \nabla \psi = \langle \overset{f}{\cancel{g}}, \psi \rangle \qquad \forall\, \psi \in \mathcal{V}(\Omega)\,.$$

We conclude using the Riesz' representation theorem.

Stokes system

$$\Updownarrow$$

$$\mu A v = P f$$

Regularity and the Stokes operator (I)

Theorem 2. Assume that $\partial\Omega$ is C^2. Then for every $f \in [L^2(\Omega)]^n$ the solution (v, p) from Theorem 1 satisfies $v \in \mathcal{H}^2(\Omega)$ and $p \in \mathcal{H}^1(\Omega)$.

Consider the Dirchlet Laplacian on Ω, i.e., $A_0 : \mathcal{D}(A_0) \to \left[L^2(\Omega)\right]^n$ defined by $\mathcal{D}(A_0) = \left[\mathcal{H}^2(\Omega)\right]^n \cap \left[\mathcal{H}_0^1(\Omega)\right]^n$ and

$$A_0\varphi = -\Delta\varphi \qquad\qquad (\varphi \in \mathcal{D}(A_0)).$$

The Stokes operator $A : \mathcal{D}(A) \to L^2_\sigma(\Omega)$ is defined by

$$\mathcal{D}(A) = \left[H^2(\Omega)\right]^n \cap \mathcal{V}(\Omega),$$

and

$$Av = PA_0v \qquad\qquad (\varphi \in \mathcal{D}(A)),$$

where P is the Leray projector.

Regularity and the Stokes operator (II)

Theorem 3. The Stokes operator A is stricly positive on $L^2_\sigma(\Omega)$ and it has compact resolvents. $(\mathcal{D}(A) \subset X$ is compact$)$

$A : \mathcal{D}(A) \longrightarrow X$, $\mathcal{D}(A)$ dense.

A symmetric if $\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle$ $\forall\, \varphi, \psi \in \mathcal{D}(A)$.

A self-adjoint if A symm. and $sI - A$ and $\bar{s}I - A$ is onto for some $s \in \mathbb{C}$.

$A > 0$ if A self-adj. and $\langle A\varphi, \varphi \rangle \geq m \|\varphi\|^2$ $\forall \varphi \in \mathcal{D}(A)$

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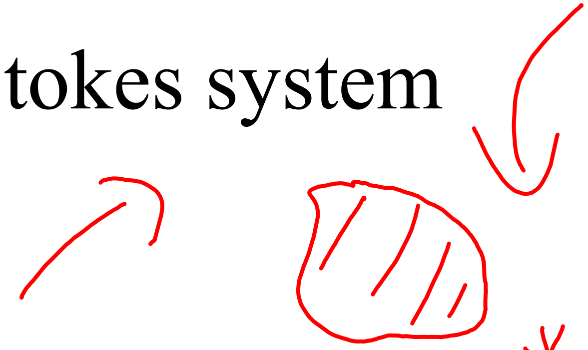
Regularity and the Stokes operator (III)

Corollary 3. There exists an orthonormal basis (φ_k) of $L^2_\sigma(\Omega)$ formed of eigenvectors of the Stokes operator A, with the corresponding sequence of eigenvalues and let (λ_k) satisfying $\lambda_k > 0$ and $\lambda_k \to \infty$. Moreover, the family $(\lambda_k^{\frac{1}{2}}\varphi_k)$ is orthonormal basis in $\mathcal{V}(\Omega)$.

Conclusions and remarks on the Stokes system



$v = v^*$

The resulsts on the Stokes system can ba adapted to

- Exterior domains;
- Non-homogeneous boundary conditions
- Domains with less smooth boundary
- Study the stationary Oseen and Navier-Stokes system.

$$-\Delta v + (v \cdot \nabla) v + \nabla p = f$$

$$\operatorname{div} v = 0, \quad v = 0 \text{ on } \partial\Omega$$