Lecture 2
An Introduction to Mathematical Analysis in Fluid Dynamics

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Semigroups in Hilbert spaces

Let X be a Hilbert space, $\mathcal{D}(A)$ a subspace of X and let $A : \mathcal{D}(A) \to X$ be a linear operator. If X is finite dimensional and $A \in \mathcal{L}(X)$, then the operators $(e^{tA})_{t\geq 0}$ describes the evolution of the state of a linear system $\dot{z}(t) = Az(t)$.

Definition 1

A family $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ of operators in $\mathcal{L}(X)$ is a *strongly continuous semigroup* on X if

(1) $\mathbb{T}_0 = I$,

(2) $\mathbb{T}_{t+\tau} = \mathbb{T}_t\mathbb{T}_\tau$ for every $t, \tau \geqslant 0$ (the semigroup property),

(3) $\lim\limits_{t\to 0,\, t>0} \mathbb{T}_t z = z$, for all $z \in X$ (strong continuity).

If $z_0 \in X$ is the initial state of the process at time $t = 0$, then its state at time $t \geqslant 0$ is $z(t) = \mathbb{T}_t z_0$. Note that $z(t+\tau) = \mathbb{T}_t z(\tau)$, so that *the process does not change its nature in time* (LTI).

Semigroups and linear evolution equations

Definition 2

The linear operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) = \left\{ z \in X \;\middle|\; \lim_{t\to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \text{ exists} \right\},$$

$$Az = \lim_{t\to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \qquad (z \in \mathcal{D}(A)),$$

is called the *infinitesimal generator* of the semigroup $\mathbb{T}$.

Proposition 1

Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. Then for every $z \in \mathcal{D}(A)$ and $t \geqslant 0$ we have that $\mathbb{T}_t z \in \mathcal{D}(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{T}_t z = A\mathbb{T}_t z = \mathbb{T}_t A z.$$

Adjoints

Let $A : \mathcal{D}(A) \to X$ be a densely defined operator. The *adjoint* of A, denoted A^*, is an operator defined on the domain

$$\mathcal{D}(A^*) = \left\{ y \in X \,\middle|\, \sup_{z\in\mathcal{D}(A),\ z\neq 0} \frac{|\langle Az, y\rangle|}{\|z\|} < \infty \right\}.$$

By the Riesz representation theorem, there exists a unique $w \in X$ such that $\langle Az, y\rangle = \langle z, w\rangle$. Then we define $A^*y = w$, so that

$$\langle Az, y\rangle = \langle z, A^*y\rangle \qquad (z \in \mathcal{D}(A),\ y \in \mathcal{D}(A^*)).$$

A is said *self-adjoint* if $A = A^*$.

Proposition 2

Let $\mathbb{T}$ be a strongly continuous semigroup on X. Then the family of operators $\mathbb{T}^* = (\mathbb{T}_t^*)_{t\geqslant 0}$ is also a strongly continuous semigroup on X, and its generator is A^*.

Checking that an operator is self-adjoint

Proposition 3

If $A_0 : \mathcal{D}(A_0) \to H$ *is symmetric,* $s \in \mathbb{C}$ *and both* $sI - A_0$ *and* $\overline{s}I - A_0$ *are onto, then* A_0 *is self-adjoint and* $s, \overline{s} \in \rho(A_0)$*.*

Example 3

Let $X = L^2[0, \pi]$ and let $A_0 : \mathcal{D}(A_0) \to X$ be the operator defined by

$$\mathcal{D}(A_0) = \left\{ z \in H^2(0, \pi) \quad | \quad z(0) = z(\pi) = 0 \right\},$$

$$A_0 z = -\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad\qquad (z \in \mathcal{D}(A_0)).$$

More examples of self-adjoint operators

Example 4 (The Dirichlet Laplacian)

$A = -A_0$, with $X = L^2(\Omega)$ and

$$\mathcal{D}(A_0) = \left\{\phi \in H_0^1(\Omega) \;\middle|\; \Delta\phi \in L^2(\Omega)\right\}, \quad A_0\phi = -\Delta\phi.$$

Example 5 (The Stokes operator)

We need more notation, such as

$$L^2_\sigma(\Omega) = \{\varphi \in L^2(\Omega;\mathbb{R}^3) \;\mid\; \operatorname{div}\varphi = 0, \;\; \varphi\cdot n = 0 \;\text{ on }\; \partial\Omega\},$$

$P : L^2(\Omega;\mathbb{R}^3) \to L^2_\sigma(\Omega)$ is called the *Leray or Helmholtz projector*

$$A_0\varphi = -\frac{\nu}{\rho}P\Delta\varphi, \qquad \mathcal{D}(A_0) = L^2_\sigma(\Omega) \cap H_0^1(\Omega;\mathbb{R}^3) \cap H^2(\Omega;\mathbb{R}^3).$$

Nonhomogeneous equations

Definition 6

With $f \in L^1_{\text{loc}}([0,\infty); X)$, consider the differential equation

$$\dot{z}(t) = Az(t) + f(t). \tag{1}$$

A *weak solution of* (1) is a function $z \in C([0,\infty); X)$ which satisfies, for every $\varphi \in \mathcal{D}(A^*)$ and every $t \geq 0$

$$\langle z(t) - z(0), \varphi \rangle = \int_0^t \left[\langle z(\sigma), A^*\varphi \rangle + \langle f(\sigma), \varphi \rangle \right] \mathrm{d}\sigma.$$

Proposition 4

With the notation of Definition 6, there exists an unique weak solution z of (1) with $z(0) = z_0 \in X$, which is given by

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} f(\sigma)\, \mathrm{d}\sigma.$$

From semigroups to evolution PDEs

When applying semigroups to evolution PDEs, X and $\mathcal{D}(A)$ are often Sobolev spaces. For instance, for heat equation in a domain $\Omega \subset \mathbb{R}^n$, one can take $X = L^2(\Omega)$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and

$$A\varphi = \Delta\varphi \qquad\qquad (\varphi \in \mathcal{D}(A)).$$

Basic question: Given a linear PDE operator A, how can one check that it generates some semigroup $\mathbb{T}$?

General answer: Apply theorems of Hille-Yosida and of Lumer-Phillips.

We limit ourselves to a consequence of these theorems, as described in the next section.

Basic de definitions

Definition 7

Let $A_0 : \mathcal{D}(A_0) \to X$ be self-adjoint. Then A_0 is *positive* if $\langle A_0 z, z\rangle \geq 0$ for all $z \in \mathcal{D}(A_0)$. A_0 is *strictly positive* if for some $m > 0$

$$\langle A_0 z, z\rangle \geqslant m\|z\|^2 \qquad (z \in \mathcal{D}(A_0)). \tag{2}$$

We say that A is negative (respectively strictly negative) and we write $A \leqslant 0$ (respectively $A < 0$) if $A = -A_0$, with A_0 positive (respectively strictly positive).

Notation. We set X_1 for $\mathcal{D}(A_0)$ endowed with the graph norm, $X_{\frac{1}{2}}$ for the completion of X_1 with respect to the norm

$$\|\varphi\|_{\frac{1}{2}} = \sqrt{\langle (I + A_0)\varphi, \varphi\rangle},$$

and $X_{-\frac{1}{2}}$ is the dual of $X_{\frac{1}{2}}$ with respect to the pivot space X.

Examples of (strictly) positive operators

Example 8 (The Dirichlet Laplacian)

$A = -A_0$, with $X = L^2(\Omega)$ and

$$\mathcal{D}(A_0) = \left\{\phi \in H_0^1(\Omega) \;\middle|\; \Delta\phi \in L^2(\Omega)\right\}, \quad A_0\phi = -\Delta\phi,$$

$$X_{\frac{1}{2}} = H_0^1(\Omega), \qquad X_{-\frac{1}{2}} = X^{-1}(\Omega).$$

Example 9 (The Stokes operator)

We need more notation, such as

$$L^2_\sigma(\Omega) = \{\varphi \in L^2(\Omega;\mathbb{R}^3) \;\mid\; \operatorname{div}\varphi = 0, \;\; \varphi\cdot n = 0 \;\text{ on }\; \partial\Omega\},$$

$P : L^2(\Omega;\mathbb{R}^3) \to L^2_\sigma(\Omega)$ is called the *Leray or Helmholtz projector*

$$A_0\varphi = -\frac{\nu}{\rho}P\Delta\varphi, \qquad \mathcal{D}(A_0) = L^2_\sigma(\Omega) \cap H_0^1(\Omega;\mathbb{R}^3) \cap H^2(\Omega;\mathbb{R}^3).$$

Semigroup generation and smoothing properties

Proposition 5

Assume that $A : \mathcal{D}(A) \to X$ *is negative. Then* A *generates a semigroup* $\mathbb{T}$ *satisfying:*

- $\|\mathbb{T}_t\| \leqslant 1$ *for every* $t \geqslant 0$,
- $\mathbb{T}_t z \in \mathcal{D}(A^\infty) \qquad\qquad (z \in X,\ t > 0).$

Proposition 6

Let $A = -A_0$, *with* $A_0 > 0$. *Then initial value problem*

$$\dot{z}(t) = Az(t) + f(t), \qquad\qquad z(0) = z_0,$$

admits, for every $z_0 \in X_{\frac{1}{2}}$ *and* $f \in L^2([0,\infty); X)$ *an unique solution* $z \in C\left([0,\infty); X_{\frac{1}{2}}\right) \cap L^2([0,\infty); X_1) \cap H^1([0,\infty); X)$.

Abstract parabolic systems with input and output

Let U, X and Y be Hilbert spaces and let A_0 be strictly positive in X. Let $A = -A_0$, $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$, $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$. Consider the system with input and output

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0, \quad y(t) = Cz(t). \tag{3}$$

Proposition 7

Let $z_0 \in X$ and the input function $u \in L^2([0,\infty); U)$. Then the system (3) *admits an unique solution $z \in C([0,\infty); X)$ with*

$$\|z(t)\|^2 + 2\int_0^t \|z(\sigma)\|^2_{\frac{1}{2}}\, \mathrm{d}\sigma = \|z_0\|^2 + 2\int_0^t \langle u(\sigma), B^* z(\sigma)\rangle_U \mathrm{d}\sigma \tag{4}$$

An abstract local existence result

Theorem 10

We assume that $Y \subset X$, with continuous and dense imbedding, that $\mathcal{N} : X \times Y \to U$ is bilinear, continuous and that there exists $K \geqslant 0$ and $p \in (0,1)$ such that

$$\|\mathcal{N}(z,y)\|_U \leqslant K\|z\|_X \, \|y\|_X^{1-p} \, \|y\|_Y^p, \qquad (z \in X, \quad y \in Y). \tag{5}$$

Moreover assume that C admits an extension $C \in \mathcal{L}(X)$ and the system is such that its output is given by $y(t) = Cz(t)$. Then or every $z_0 \in X$ and every $u \in L^2([0,\infty);U)$, there exists $\tau > 0$ such that the system (3)

$$\dot{z}(t) = Az(t) + B\mathcal{N}(z(t),y(t)), \quad z(0) = z_0, \tag{6}$$

admits a solution $z \in C([0,\tau];X)$.

Idea of the proof (I)

Let $T > 0$ and let $\mathcal{G}_T : L^2([0,T];U) \to L^2([0,T];U)$ be defined by

$$[\mathcal{G}_T(v)](t) = \mathcal{N}(z(t), y(t)) \qquad \text{for} \quad t \in [0,T], \quad v \in L^2([0,T];U),$$

where

$$\dot{z}(t) = Az(t) + Bv(t), \qquad z(0) = z_0.$$

To that show $\mathcal{G}_T$ has a fixed point we note that, from (5),

$$\|\mathcal{G}_T(v)(t)\|_U^{\frac{2}{p}} \leqslant K^{\frac{2}{p}} \|C\|_{\mathcal{L}(X)}^{\frac{2(1-p)}{p}} \|z(t)\|_X^{\frac{4-2p}{p}} \|y(t)\|_Y^2.$$

On the other hand, there exists $k_T > 0$ with

$$\|z(t)\| + \|y\|_{L^2([0,T];Y)} \leqslant k_T \left(\|z_0\| + \|v\|_{L^2([0,T];U)}\right).$$

Consequently:

Idea of the proof (II)

$$\|\mathcal{G}_T(v)\|^{\frac{2}{p}}_{L^{\frac{2}{p}}([0,T];U)} \leqslant k_T\left(\|z_0\| + \|v\|_{L^2([0,T];U)}\right)^{\frac{4}{p}}. \quad (7)$$

Let $B_{M,T}$ be the ball of radius M, centered at the origin, in $L^2([0,T];U)$. From (7) it follows that for $v \in B_{M,T}$ we have

$$\|\mathcal{G}_T(v)\|_{L^{\frac{2}{p}}([0,T];U)} \leqslant k_T^{\frac{p}{2}}(2M)^2. \quad (8)$$

Applying Hölder's inequality, we obtain that for every $v \in B_{M,T}$ we have

$$\|\mathcal{G}_T(v)\|_{L^2([0,T];U)} \leqslant k_T^{\frac{p}{2}}(2M)^2\, T^{\frac{1-p}{2}}.$$

For T sufficiently small (depending on the system and on M) we have $\mathcal{G}_T(v) \in B_{M,T}$ for every $v \in B_{M,T}$.
We end up by showing the contraction property.

The viscous Burgers equation

$$\begin{cases} \dot{z}(t,x) = z_{xx}(t,x) - z(t,x)z_x(t,x) & t \geqslant 0,\ x \in (0,1), \\ z(t,0) = z(t,1) = 0 & t \geqslant 0, \\ z(0,x) = z_0(x) & y \in (0,1), \\ y(t,x) = z(t,x) & t \geqslant 0,\ x \in (0,1). \end{cases} \tag{9}$$

Theorem 11

For every $z_0 \in H_0^1(0,1)$ *there exists a unique solution* z *of* (9) *such that*

$$z \in H^1_{\mathrm{loc}}((0,\infty); L^2[0,1]) \cap C([0,\infty); H_0^1(0,1)) \cap L^2_{\mathrm{loc}}\left([0,\infty); H^2(0,1)\right).$$

Proof (I): Local in time solutions

We introduce the state, input and output spaces

$$X = H_0^1(0,1), U = L^2(0,1), \quad Y = H^2(0,1) \cap H_0^1(0,1),$$

and the strictly negative operator $A : \mathcal{D}(A) \to X$ by

$$A\varphi = \varphi_{xx}, \qquad \varphi \in \mathcal{D}(A) = \left\{\varphi \in H^3(0,1) \ \middle|\ \varphi,\ \varphi_{xx} \in H_0^1(0,1)\right\}.$$

In this case $X_{\frac{1}{2}} = H^2(0,1) \cap H_0^1(0,1), \quad X_{-\frac{1}{2}} = L^2[0,1]$. We take B the identity operator on $L^2[0,1]$. We denote by C the identity operator of $H_0^1(0,1)$, which can be restricted to an unbounded observation operator $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$.
Define $\mathcal{N} : X \times Y \to U$ is defined by $\mathcal{N}(z,y) = -zy_x$. We have

$$\|\mathcal{N}(z,y)\|_U = \|zy_x\|_{L^2[0,1]} \leqslant \|z\|_{C[0,1]} \|y_x\|_{L^2[0,1]} \leqslant K_0\|z\|_X\|y\|_X,$$

so that (5) holds with $p = 0$ and thus for every any $p \in (0,1)$.

Proof (II): Global in time solutions

First note that we have the energy estimates

$$\|z(t)\|_{L^2}^2 + \int_0^t \|z_x(\sigma)\|_{L^2}^2 \,\mathrm{d}\sigma \leqslant \|z_0\|_{L^2}^2.$$

$$\|z_x(t)\|_{L^2}^2 + \int_0^t \|z_{xx}(\sigma)\|_{L^2}^2 \,\mathrm{d}\sigma \leqslant K \left[\|(z_0)_x\|_{L^2}^2 + \int_0^t \|z(\sigma) z_x(\sigma)\|_{L^2}^2 \,\mathrm{d}\sigma \right].$$

Note that
$\|z(t)\, z_x(t)\|_{L^2} \leqslant \|z(t)\|_{C[0,1]} \|z_x(t)\|_{L^2} \leqslant \sqrt{2} \|z(t)\|_{L^2}^{\frac{1}{2}} \|z_x(t)\|_{L^2}^{\frac{3}{2}}$. It follows that

$$\|z_x(t)\|_{L^2}^2 \leqslant K_1 + K_2 \int_0^t (\|z_x(\sigma)\|_{L^2}) \|z_x(\sigma)\|_{L^2}^2 \,\mathrm{d}\sigma,$$

and we conclude by Gronwall's inequality.

The incompressible Navier-Stokes equations

$$\rho \dot{z} - \nu \Delta z + \rho (z \cdot \nabla) z + \nabla p = u_1, \qquad t \geqslant 0, \; x \in \Omega, \tag{10}$$

$$\mathrm{div} z = 0, \qquad t \geqslant 0, \; x \in \Omega, \tag{11}$$

$$z = 0, \qquad t \geqslant 0, \; x \in \partial\Omega, \tag{12}$$

$$z(0,x) = z_0(x), \qquad x \in \Omega. \tag{13}$$

Theorem 12

For every initial state $z_0 \in H_0^1(\Omega; \mathbb{R}^3)$ *with* $\mathrm{div} z_0 = 0$*, there exists* $T > 0$ *and a unique solution* (z, p) *of* (10)-(13) *such that*

$$z \in C([0,T]; H_0^1(\Omega)) \cap L^2([0,T]; L^2(\Omega), \quad p \in L^2([0,T]; \widehat{H^1}(\Omega)).$$

Idea of the proof

Take $X = V(\Omega)$, $Y = H^2(\Omega) \cap V(\Omega)$, $U = L^2(\Omega)$, and let $A = -A_0$, where A_0 is the part of the Stokes operator in X. B and C are the identity of $L^2(\Omega)$ and $N : X \times Y \to U$ is defined by Define $N(z, y) = -P[(z \cdot \nabla)y]$. To estimate N we note that

$$\int_\Omega z_i^2(x) \left[\frac{\partial y_j}{\partial x_i}(x)\right]^2 \mathrm{d}x = \int_\Omega z_i^2(x) \left[\frac{\partial y_j}{\partial x_i}(x)\right]^{2/5} \left[\frac{\partial y_j}{\partial x_i}(x)\right]^{8/5}$$

$$\leqslant \|z_i\|^2_{L^5(\Omega)} \left\|\frac{\partial y_j}{\partial x_i}\right\|^{2/5}_{L^2(\Omega)} \left\|\frac{\partial y_j}{\partial x_i}\right\|^{8/5}_{L^4(\Omega)}.$$

Since $H^1 \subset L^6$, $\int_\Omega z_i^2(x) \left[\frac{\partial y_j}{\partial x_i}(x)\right]^2 \leq K\|z_i\|^2_{H_0^1}\|y_i\|^{2/5}_{H_0^1}\|y_i\|^{8/5}_{H^2}$.

Thus $\|(z \cdot \nabla)y\|_U \leqslant \widetilde{K}\|z\|_X\|y\|_X^{1/5}\|y\|_Y^{4/5}$ so (5) holds with $p = 4/5$.