PDE systems describing the motion of solids in a viscous fluid:

wellposedness, control and long-time behavior

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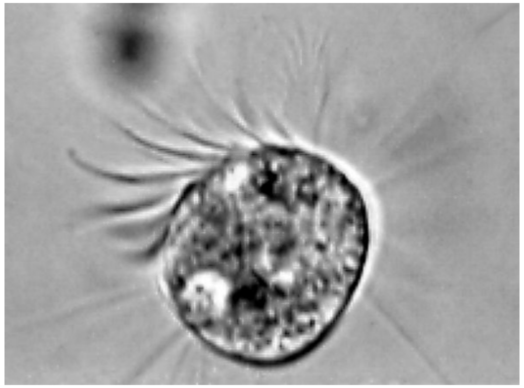
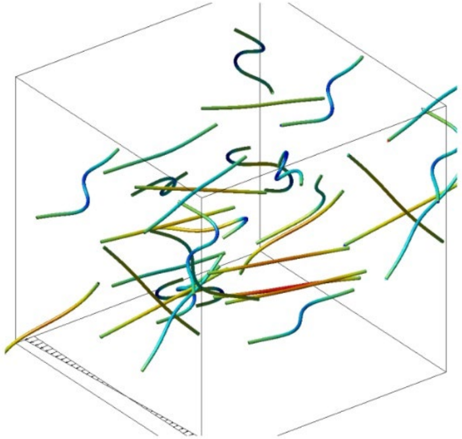
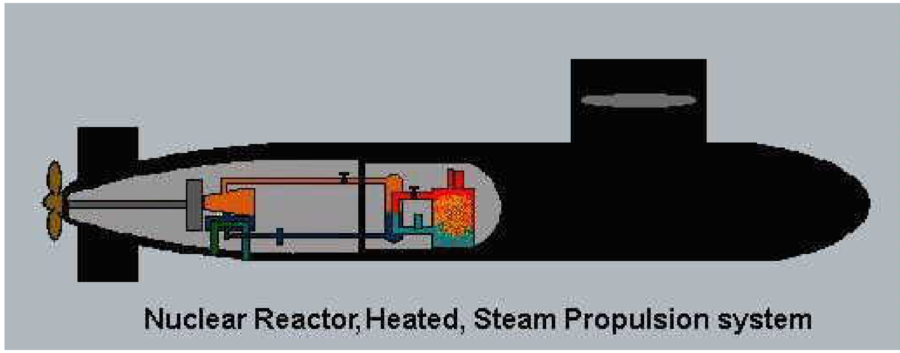
Fluid-Structure Interactions

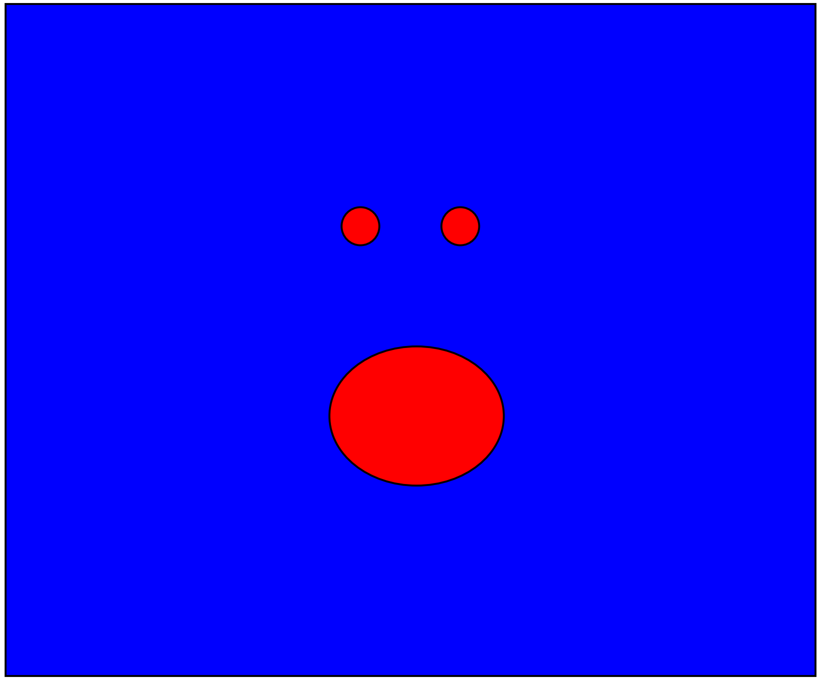
Deformable or rigid structures interact with perfect, Newtonian or non-Newtonian fluids in : biology, meteorology, geology, aerospace and chemical engineering, … .







Nonlinear free boundary problems



- Euler or Navier-Stokes equations for the fluid
- ODE's system for the rigid bodies motion
- Continuity of the velocity field
- Homogenous Dirichlet boundary conditions on the exterior boundary
- Free boundary problem

First wellposedness results (even with rigids solids) around year 2000 !

[1] D. Serre (1987) ,
[2] K. H. Hoffmann and V. Starovoitov (1998, 2000)
[3] B. Desjardins and M. Esteban (1999,2000)
[4] C. Conca, J. San Martin and M. Tucsnak (2001)
[5] J. San Martin, V. Starovoitov and M. Tucsnak (2002)
[6] Takahashi (2003)

Mathematical Challenges

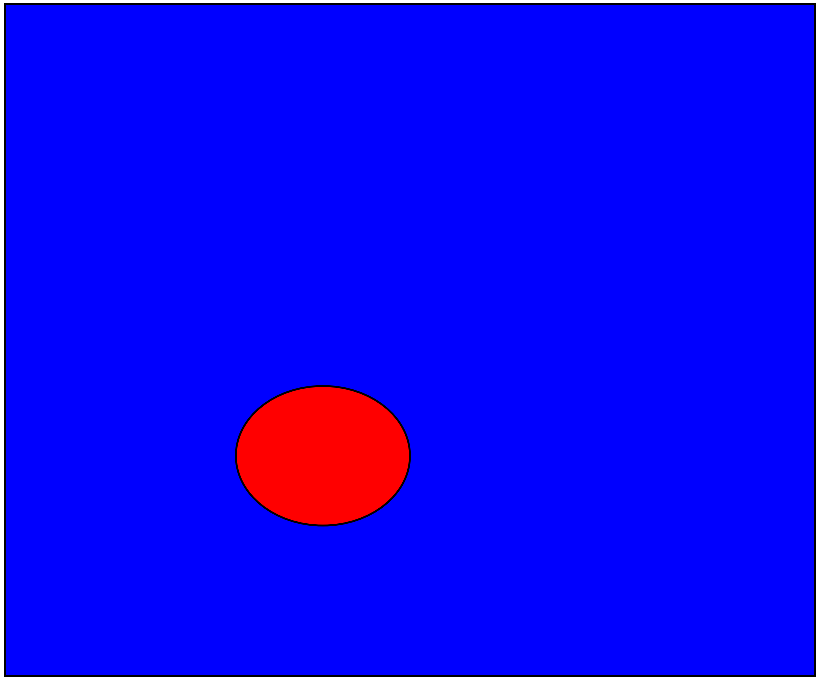
- Existence and uniqueness
  - Free boundary
  - Possibility of contacts

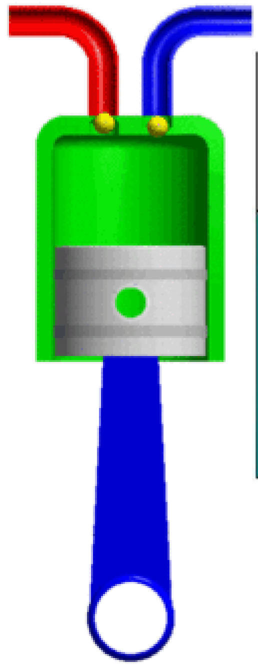
- Control

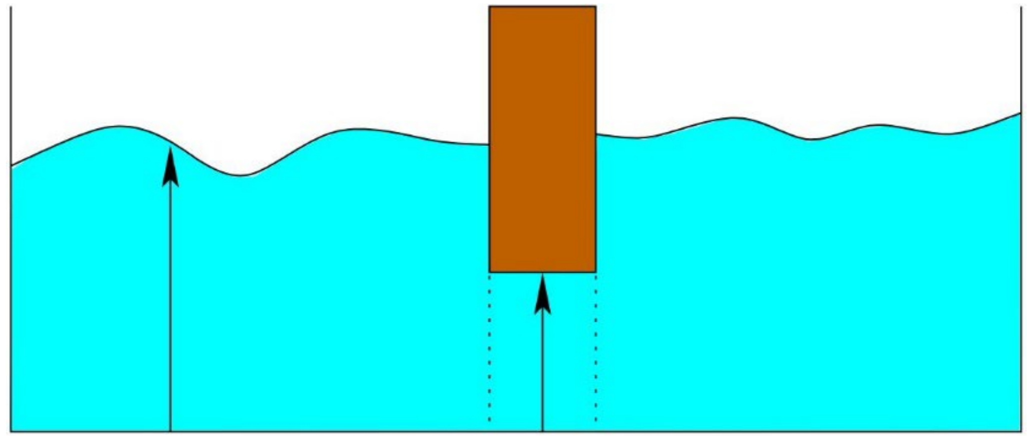
- Long-time behavior

Solids moving in a fluid:

Do they (asymptotically) stop ?







Natural question: will the solid stop when time tends to infinity?

Outline

- Modelling and wellposedness
  - Rigid immersed solids
  - Fish-like swimming
- A control problem : low Reynolds number swimming
- Long-time behavior
- The point absorber problem

Modelling and wellposedness

Governing equations for a disk moving in an incompressible fluid

$$\rho\frac{\partial v}{\partial t} - \nu\Delta v + \rho(v\cdot\nabla)v + \nabla p = 0 \qquad \text{in } \mathcal{F}(h(t)),\ t\in(0,T),$$

$$\text{div } v = 0 \qquad \text{in } \mathcal{F}(h(t)),\ t\in(0,T),$$

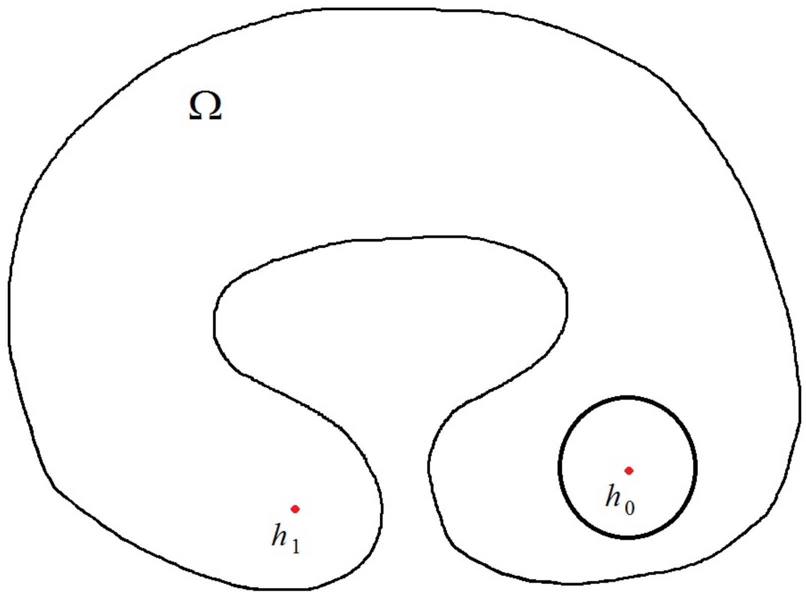
$$v = 0 \qquad \text{on } \partial\Omega,\ t\in(0,T),$$

$$v = \dot h + \omega\,(x-h)^{\perp} \qquad \text{on } \partial\mathcal{B}(h(t)),\ t\in(0,T),$$

$$m\ddot h = -\int_{\partial\mathcal{B}(h(t))}\sigma(v,p)n\,\mathrm{d}\Gamma + u$$

$$I\dot\omega = -\int_{\partial\mathcal{B}(h(t))}(x-h)^{\perp}\cdot\sigma(v,p)n\,\mathrm{d}\Gamma$$

$$v(0,x) = v_0(x), h(0) = h_0, \dot h(0) = g_0, \omega(0) = \omega_0,$$



Energy estimate (I)

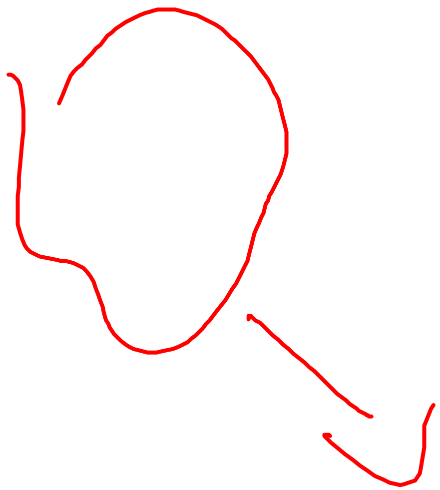
$$E(t) = \frac{1}{2}\int_{\mathcal{F}} \rho v^2 \, dx + \frac{1}{2} m |\dot{h}|^2 + \frac{1}{2}\gamma |w|^2$$

Energy estimate (II)

$$\dot{E}(t) = -\nu \int_{\Omega} |\nabla v|^2$$

Energy estimate (III)

Potential flows and Kirchhoff's equations (I)



$$\Delta \phi = 0$$

$$\frac{\partial \phi}{\partial n} = V_R \cdot n$$

Potential flows and Kirchhoff's equations (II)

Potential flows and Kirchhoff's equations (III)

Potential flows and Kirchhoff's equations (IV)

What is a strong solution?

Definition.

A quadruple (h, ω, v, p) is called a strong solution on $[0, T]$ if

1. $h \in H^2(0, T)$, $\omega \in H^1(0, T)$ and $d(h(t), \partial\Omega) < 1$ for every $t \in [0, T]$.

2. $v \in C([0, T]; H_0^1(\Omega))$, $p \in L^2([0, T]; L^2(\Omega))$ and

$$v(t, x) = \dot{h}(t) + \omega(t)(x - h(t))^\perp \qquad (t \geqslant 0, \quad x \in \mathcal{B}(h(t))).$$

3. For every map $X \in C^1([0, T]; H_0^1(\Omega)) \cap L^2([0, T]; H^2(\mathcal{F}(h_0)))$, where, for $t \in [0, T]$, the map $X(t, \cdot)$ is H^1 invertible from Ω onto Ω and from $\mathcal{F}(h_0)$ onto $\mathcal{F}(h(t))$ we have

$$v \circ X \in L^2([0, T]; H^2(\mathcal{F}(h_0))), \qquad p \circ X \in L^2([0, T]; H^1(\mathcal{F}(h_0)))$$

4. The equations are satisfied in the strong sense (the terms are in L^2).

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Relation with Lagrangian and ALE Formulations

To construct X we consider a smooth enough vector field $z(t,x)$ such that

$$z = 0 \text{ on } \partial\Omega, \quad z = \dot{h}(t) + \omega(t)(x - h(t))^{\perp} \text{ in } \mathcal{B}(h(t)),\, t \in (0,T),$$

and we consider the initial value problem

$$\frac{\partial X}{\partial t}(t,y) = z(t, X(t,y)), \quad X(0,y) = y \in \Omega.$$

- Such transformations X appear in the Arbitrary Lagrangian Eulerian (ALE) formulations of the Navier-Stokes system
- If we take $z = v$ then

  $v \circ X$ and $p \circ X$ satisfy the *Lagrangian formulation of the system.*

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Some references

- *Grandmont and Maday (M2AN, 2000) :*

  Local in time existence and uniqueness using Lagrange formulation

- *Takahashi and Tucsnak (JMFM, 2004) :*

  2D motion of a disk-viscous fluid system filing the whole space

- *Takahashi (Adv. Diff. Eq., 2003),* Motion of a rigid-fluid system in a bounded domain

- *Hillairet and Takahashi, (Ann. IHP, 2010):*

  Global existence (based on lack of collisions) in a half-plane

- *Geissert, Götze and Hieber (TAMS, 2013)* : L^p-theory in 3D

  Ervedoza, Maity and Tucsnak (Math. Ann., 2023)

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Change of variables (I)

We first construct a vector field $z(x,t)$ such that

$$\text{div } z = 0 \quad \text{in } \mathbb{R}^2 \setminus \mathcal{S}(t),\ t \in (0,T),$$

$$z = 0 \quad \text{on } \partial\Omega,\ t \in (0,T), \qquad z = u \quad \text{in } \mathcal{S}(t),\ t \in (0,T).$$

with

$$\left|\frac{\partial z}{\partial t}(x_1,x_2,t)\right| + \left|\frac{\partial^{\alpha_1+\alpha_2} z}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}(x_1,x_2,t)\right| \leq C_0\left(1 + |\dot{h}(t)| + |\dot{\theta}(t)|\right).$$

Consider the initial value problem

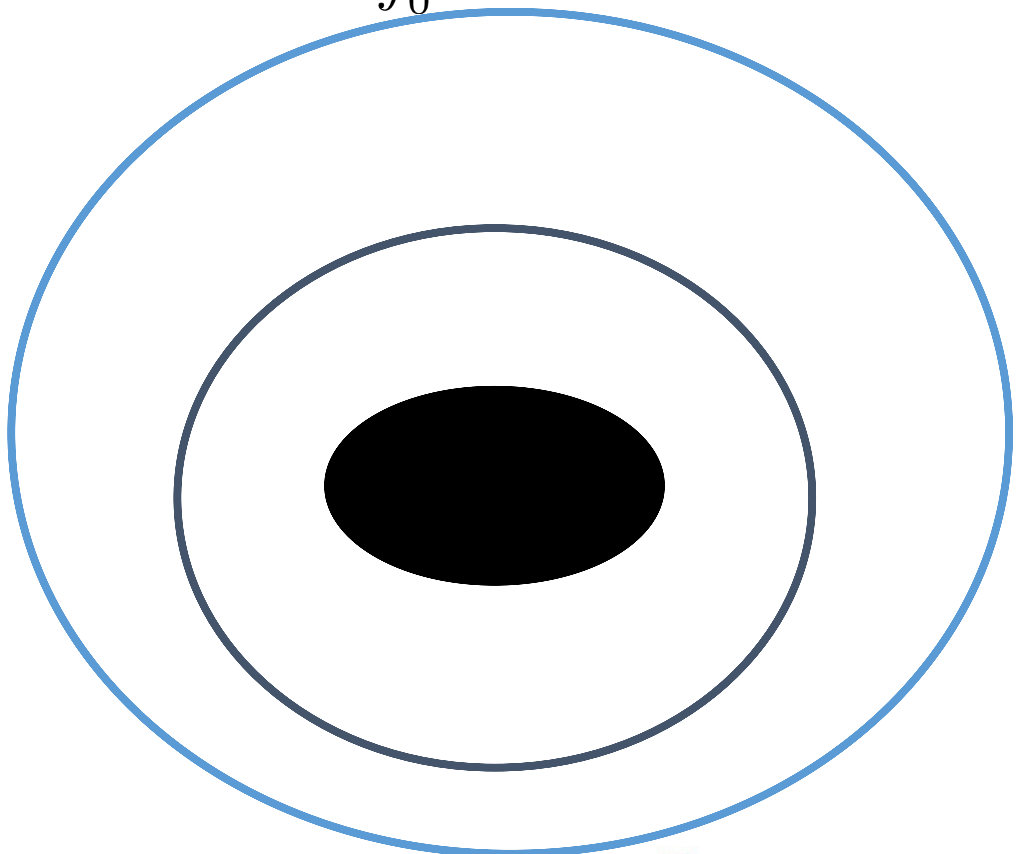
$$\frac{\partial X}{\partial t}(y,t) = z(X(y,t),t), \qquad X(y,0) = y \in \mathbb{R}^2.$$

Change of variables (II)

$$U(y,t) = J_Y(X(y,t),t)v(X(y,t),t), \quad P(y,t) = p(X(y,t),t),$$

$$H(t) = \int_0^t R_{\theta(s)}\dot{h}(s)\,\mathrm{d}s.$$

$X = I$



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The transformed system

$$
\begin{aligned}
\frac{\partial U}{\partial t} + [MU] - \nu[LU] + [NU] + [GP] &= 0, && \text{in } \mathcal{F} \times (0,T),\\
\operatorname{div} \; U &= 0, && \text{in } \mathcal{F} \times (0,T),\\
U &= 0, && \text{on } \partial\Omega \times (0,T),\\
U(y,t) &= \dot{H}(t) + \dot{\theta}(t) y^{\perp}, && y \in \partial\mathcal{S},\; t \in (0,T),\\
M\ddot{H}(t) - M\dot{\theta}(t)\dot{H}(t)^{\perp} &= -\int_{\partial\mathcal{S}} \Sigma(U,P)n \,\mathrm{d}\Gamma && t \in (0,T),\\
\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) &= -\int_{\partial\mathcal{S}} y^{\perp} \cdot \Sigma(U,P)n \,\mathrm{d}\Gamma && \text{in } (0,T).
\end{aligned}
$$

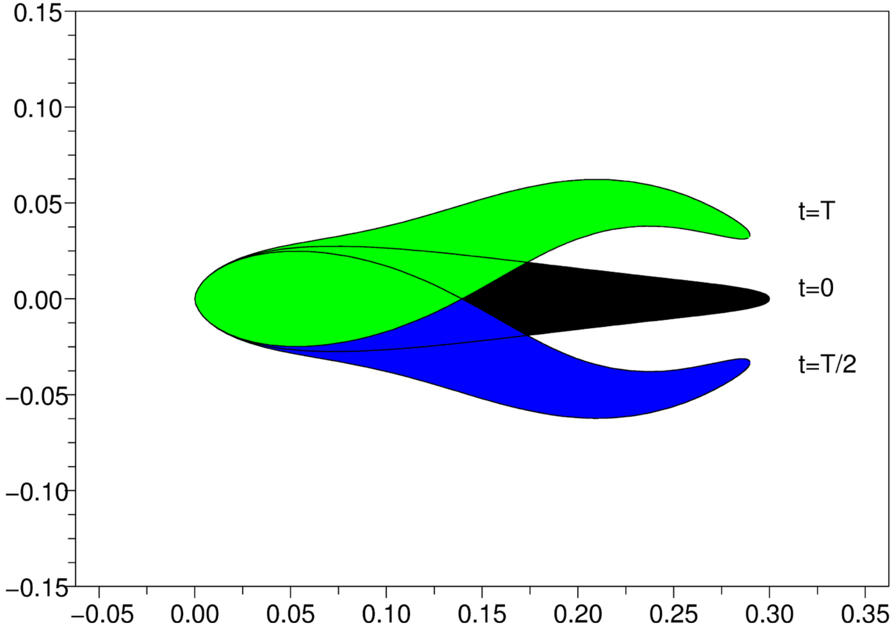
Main steps of the proof

- Estimate the coefficients in M, L, N, G as functions of H and θ.

- Use the above estimates in a fixed point procedure to get local in time existence of strong solutions

- Go back to the physical space and use energy estimates to get global existence in 2D.

Kinematics of fish-like swimming



(H1) $\forall t \geq 0$, the map $y \mapsto X^*(y,t)$ is a C^∞ diffeomorphism from $\mathcal{S}_0$ onto $\mathcal{S}^*(t)$.

(H2) $\forall t \geq 0, \quad \int_{\mathcal{S}^*(t)} \mathrm{d}x^* = \int_{\mathcal{S}_0} \mathrm{d}y.$

(H3) $\forall t \geq 0, \quad \displaystyle\int_{\mathcal{S}^*(t)} \rho^*(x^*,t) w^*(x^*,t)\,\mathrm{d}x^* = 0$

where $w^*(x^*,t) = \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t)$.

(H4) $\forall t \geq 0, \quad \displaystyle\int_{\mathcal{S}^*(t)} \rho^*(x^*,t)\; x^{*\perp} \cdot w^*(x^*,t)\,\mathrm{d}x^* = 0 \quad .$

The governing equations

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla p = 0 \qquad \text{in } \mathcal{F}(\xi,\theta,t),\ t \in (0,T),$$

$$\text{div } v = 0 \qquad \text{in } \mathcal{F}(\xi,\theta,t),\ t \in (0,T),$$

$$v = 0 \qquad \text{on } \partial\Omega,\ t \in (0,T),$$

$$v = \dot{\xi} + \dot{\theta}\,(x-\xi)^{\perp} + w \qquad \text{on } \partial\mathcal{S}(\xi,\theta,t),\ t \in (0,T),$$

$$m\ddot{\xi} = -\int_{\partial\mathcal{S}(\xi,\theta,t)} \sigma(v,p)n \,\mathrm{d}\Gamma \qquad \text{in } (0,T),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) = -\int_{\partial\mathcal{S}(\xi,\theta,t)} (x-\xi)^{\perp} \cdot \sigma(v,p)n \,\mathrm{d}\Gamma \qquad \text{in } (0,T).$$

Two wellposedness results

Theorem 1 (J. San Martin, J. –F. Scheid, T. Takahashi and M.T., ARMA(2008)).
In two space dimensions, if the given deformation is smooth enough than the system admits an unique strong solution. If no contact occurs in finite time then this solution is global.
Proof. Change of variables+Fujita-Kato semigroup approach

Theorem 2 (S. Necasova, T. Takahashi and M.T., Acta Appl Math (2011)).
In two or three space dimensions, the system admits at least a weak global in time solution (up to possible collisions)
Proof. Adaptation of the penalty method introduced in
Starovoitov, San Martin and M.T., ARMA 2002.

A control problem :
low Reynolds number swimming

G. I. Taylor’s movie (1956)

High Reynolds | **Low Reynolds**

A system with a geometric input

$u = X^*$

Navier-Stokes
Euler
Stokes
Inviscid fluid

$y = (\xi, \dot{\xi}, \theta, \dot{\theta})$

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Dynamics (I)

$$\begin{cases} -\mu\Delta \boldsymbol{v} + \nabla p &= 0 \quad \text{in } \mathbb{R}^3 \setminus S \\ \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in } S \end{cases}, \qquad \lim_{|x|\to\infty} \boldsymbol{v}(\mathbf{x}, t) = 0$$

$$\boldsymbol{v}(x,t) = \dot{\xi}(t) + \omega(t) \times (x - \xi(t)) + R(t)\frac{\partial X^*}{\partial t}(R^*(t)(\mathbf{x} - \xi(t)), t) \quad (x \in \partial S),$$

$$0 = \int_{\partial S} \sigma \mathbf{n} \qquad 0 = \int_{\partial S} (x - \xi(t)) \times \sigma \mathbf{n} \, \mathrm{d}\Gamma,$$

$$\sigma = \sigma(\boldsymbol{v}, p) = \mu\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\right) - pI_3.$$

The above system reduces to

$$\mathcal{F}_1(\xi(t), R(t); X^*) \begin{bmatrix} \dot{\xi} \\ \omega \end{bmatrix} + \mathcal{F}_2(\xi(t), R(t); X^*) \frac{\partial X^*}{\partial t}(\cdot, t) = 0,$$

where:

Dynamics (II)

$\mathcal{F}_1(S)\begin{bmatrix} g \\ \omega \end{bmatrix} = \int_{\partial\Omega} \sigma(\widetilde{v}, \widetilde{p})\mathbf{n}$, with

$$\begin{cases} -\mu\Delta\widetilde{v} + \nabla\widetilde{p} &= & 0 & \text{in } \mathbb{R}^3 \setminus S \\ \operatorname{div}\widetilde{v} &= & 0 & \text{in } \mathbb{R}^3 \setminus S \end{cases}$$

$$\lim_{|x|\to\infty} \widetilde{v}(\mathbf{x}) = 0, \quad \widetilde{v}(x,t) = g(t) + \omega(t) \times (x - \xi(t)) \quad (x \in \partial S),$$

$\mathcal{F}_2(S)(\xi(t), R(t); X^*(\cdot, t)) = \int_{\partial\Omega} \sigma(\widehat{v}, \widehat{p})\mathbf{n}$, with

$$\begin{cases} -\mu\Delta\widehat{v} + \nabla\widehat{p} &= & 0 & \text{in } \mathbb{R}^3 \setminus S \\ \operatorname{div}\widehat{v} &= & 0 & \text{in } \mathbb{R}^3 \setminus S \end{cases}$$

$$\lim_{|x|\to\infty} \widehat{v}(\mathbf{x}) = 0, \quad \widehat{v}(x,t) = R(t)\frac{\partial X^*}{\partial t}(R^*(t)(\mathbf{x} - \xi(t)), t) \quad (x \in \partial S).$$

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Dynamics (III) :
governing equations and scallop theorem

The governing equations simply write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ R\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & A(\omega)\end{bmatrix}\left[\mathcal{F}_1(\xi(t),R(t);X^*)\right)]^{-1}\mathcal{F}_2(\xi(t),R(t);X^*)\frac{\partial X^*}{\partial t}.$$

Scallop Theorem (Purcell, 1977):

Assume that the deformation X^* is reciprocal, i.e. that $X(\cdot,t) = X(\cdot,g(t))$, where $g:[0,\tau]\to[0,\tau]$, with $g(0)=g(\tau)=0$ is a Lipschitz continuous function. Then $\xi(\tau)=\xi(0)$ and $R(\tau)=R(0)$.

The control system (axially symmetric case)

The equilibrium condition gives $\mathcal{F}_1(X^*(t,\cdot))\dot{\xi} + \mathcal{F}_2(X^*(t,\cdot))\frac{\partial X^*}{\partial t}(\cdot,t) = 0$, or

$$\dot{\xi}(t) = \mathcal{F}(\xi(t); X^*(\cdot,t))\frac{\partial X^*}{\partial t}(\cdot,t).$$

Assume that $X^*(x,t) = \sum_{j=1}^{N} \alpha_j(t) D_j(x)$. Then

$$\dot{\xi}(t) = \sum_{j=1}^{N} \dot{\alpha}_j(t) \underbrace{\mathcal{F}(\alpha(t)) D_j}_{f_j(\alpha(t))}.$$

Reformulation as a bilinear control system

$$\dot{\xi}(t) = \sum_{j=1}^{N} u_j(t)\, f_j(\alpha(t)), \qquad \dot{\alpha}(t) = u(t).$$

The above equation determine a control system, of state

$$z(t) = \begin{bmatrix} \xi(t) \\ \alpha(t) \end{bmatrix} \in \mathbb{R}^{N+1}$$

and of control function $u \in L^\infty([0,\infty);\mathbb{R})$. Setting $F_j(z) = \begin{bmatrix} f_j(\alpha) \\ 1 \end{bmatrix}$, the system writes

$$\dot{z}(t) = \sum_{j=1}^{N} u_j(t)\, F_j(z(t)),$$

i.e. we have a *bilinear control system.*

Radial deformations (I)

Take S_0 to be the unit ball in $\mathbb{R}^3$ and

$$X^\star(t,\mathbf{y}) = \big(1 + r^\star\big(t, \cos\theta(\mathbf{y})\big)\big)\,\mathbf{y} \qquad (t \in [0,T],\ \mathbf{y} \in S_0)\,,$$

We consider deformations of the form du type

$$X^\star(t,\mathbf{y}) = \mathbf{y} + \alpha_1(t) D_1(\mathbf{y}) + \alpha_2(t) D_2(\mathbf{y}) \qquad ((t,\mathbf{y}) \in [0,T] \times S_0)\,,$$

where

$$D_i(\mathbf{y}) = P_{i+1}\big(\cos\theta(\mathbf{y})\big)\mathbf{y} \qquad (i \geqslant 1\,,\ \mathbf{y} \in S_0)\,,$$

and $(P_i)_{i\in\mathbb{N}}$ are the Legendre polynomials.

The above assumptions allow almost explicit calculations for the Stokes equations.

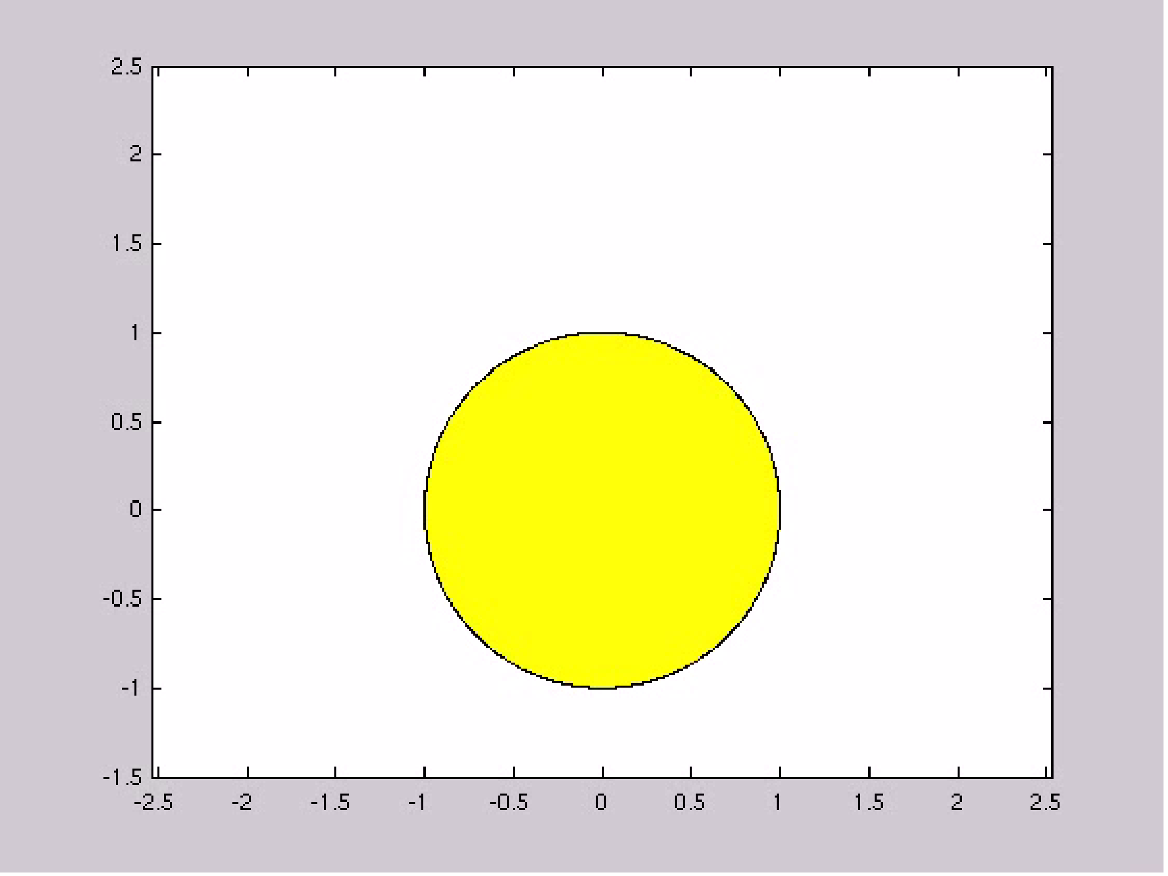
Radial Deformations (II):
Controllability and time optimal control

Theorem (Lohéac and M.T., 2013). For every $N \geqslant 2$ and $\xi_1 \in \mathbb{R}$, there exists a minimal $T > 0$ and $u \in C^\infty\left([0,T], \mathbb{R}^N\right)$ such that the solution (ξ, α) of (1) satisfies

1. $\xi(T) = \xi_1$ and $\alpha(T) = 0$,
2. for every $t \in [0,T]$, $|u(t)|_2 \leqslant 1$,
3. for every $t \in [0,T]$, $|\alpha(t)|_1 < 1$.

Idea of the proof. We can apply Chow's theorem, provided that we have information of the derivatives of F_j with repect to α. (shape differention) We end up with a compactness argument.

Radial Deformations (III):
a simulation of time optimal trajectories



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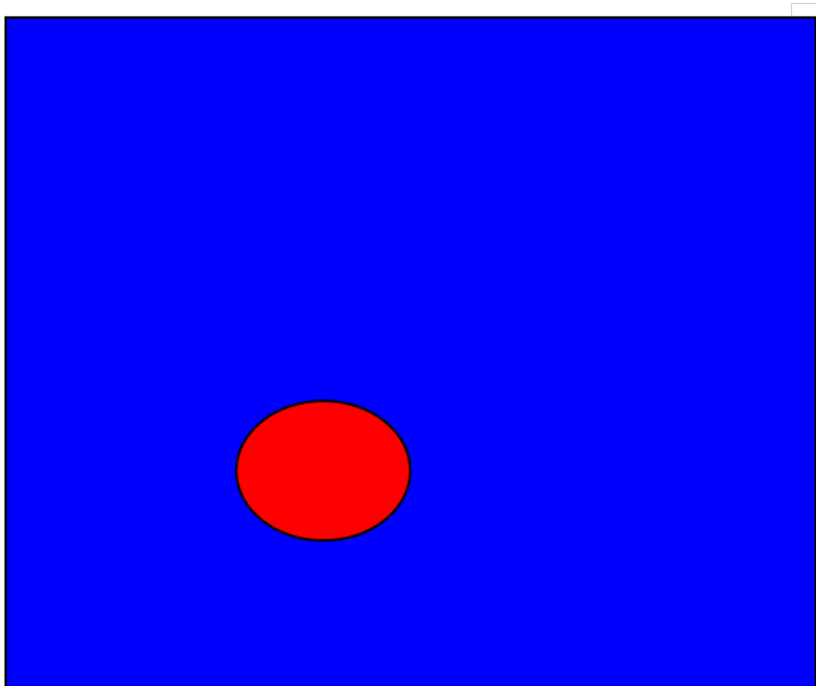
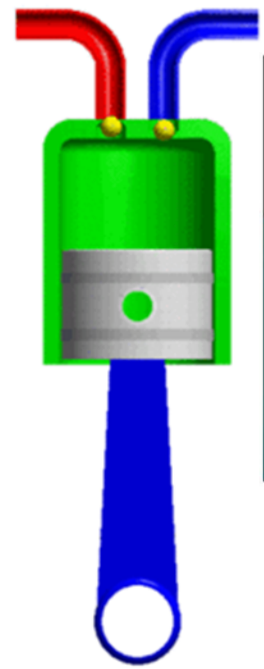
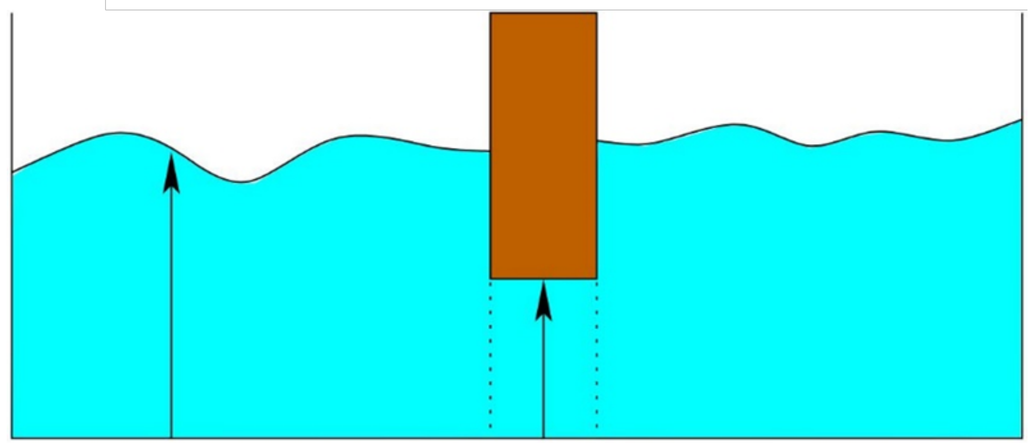
Long-time behavior

Solids moving in a fluid:
Do they (asymptotically) stop ?







Natural question: will the solid stop when time tends to infinity?

The governing equations

$$
\begin{aligned}
\rho\frac{\partial v}{\partial t} - \nu\Delta v + \rho(v\cdot\nabla)v + \nabla p = 0 \qquad & \text{in } \mathcal{F}(h(t)),\ t\in(0,T),\\
\text{div } v = 0 \qquad & \text{in } \mathcal{F}(h(t)),\ t\in(0,T),\\
v = 0 \qquad & \text{on } \partial\Omega,\ t\in(0,T),\\
v = \dot h + \omega\times(x-h) \qquad & \text{on } \partial\mathcal{B}(h(t)),\ t\in(0,T),\\
m\ddot h = -\int_{\partial\mathcal{B}(h(t))}\sigma(v,p)n\,\mathrm{d}\Gamma \qquad & \text{in } (0,T),\\
J\dot\omega = -\int_{\partial\mathcal{B}(h(t))}(x-h)\times\sigma(v,p)n\,\mathrm{d}\Gamma \qquad & \text{in } (0,T),\\
\&\\
v(x,0)=v_0(x),\ h(0)=h_0,\ \dot h(0)=g_0,\ \omega(0)=\omega_0, \qquad & .
\end{aligned}
$$

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Context and main result (for "small" data)

Existing results:

- Exponential decay of the velocity for bounded Ω (Takahashi, 2003);
- $\sup_{t>0} t^{\frac{1}{2}} |\dot{h}(t)| < \infty$ for $\Omega = \mathbb{R}^2$ (Ervedoza, Hillairet, Lacave (2013));
- Nothing known for $\Omega = \mathbb{R}^3$ or for $\Omega = \mathbb{R}^2$ and a solid which is not a disk.

Theorem. (Ervedoza, Maity and M.T., Math. Annalen+JMFM, 2023)
For $\Omega = \mathbb{R}^3$, $v_0 \in W^{1,p}$ for every $p > 1$ and $\|v_0\|_{L^3} + |g_0| + |\omega_0| << 1$ we have

$$\sup_{t>0} t^{\frac{3}{2}} |\dot{h}(t)| < \infty.$$

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Linearization (I)

$$
\begin{aligned}
\rho\frac{\partial v}{\partial t} - \nu\Delta v + \nabla p = 0 \qquad & \text{in } \mathcal{F}(h_0),\ t \in (0,T),\\
\text{div } v = 0 \qquad & \text{in } \mathcal{F}(h_0),\ t \in (0,T),\\
v = 0 \qquad & \text{on } \partial\Omega,\ t \in (0,T),\\
{\color{blue} v = g + \omega \times (x - h_0)} \qquad & \text{on } \partial\mathcal{B}(h_0),\ t \in (0,T),
\end{aligned}
$$

$$
\begin{aligned}
m\dot{g} = -\int_{\partial\mathcal{B}(h_0)} \sigma(v,p)n \, \mathrm{d}\Gamma \qquad & \text{in } (0,T),\\
J\dot{\omega} = -\int_{\partial\mathcal{B}(h_0)} (x - h_0) \times \sigma(v,p)n \, \mathrm{d}\Gamma \qquad & \text{in } (0,T),\\
\&\\
v(x,0) = v_0(x),\ \ g(0) = g_0, \omega(0) = \omega_0, \qquad & .
\end{aligned}
$$

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Linearization (II)

Let (v, g, ω) satisfy the linearized system. We set

$$z(t,x) = \begin{cases} v(t,x) & (x \in \mathbb{R}^3 \setminus \mathcal{O}), \\ g(t) + \omega(t) \times (x - h_0) & (x \in \mathcal{O}). \end{cases}$$

For $\varphi \in H^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3 \setminus \mathcal{O})$ we set $D(\varphi) = \frac{1}{2}(\nabla\varphi + (\nabla\varphi)^*)$. Assuming that $D(\varphi) = 0$ in $\mathcal{O}$ and that $\operatorname{div}\varphi = 0$ in $\mathbb{R}^3$, we set

$$\mathcal{A}_2\varphi = \begin{cases} \mu\Delta\varphi & \text{in } \Omega, \\ 2\mu m^{-1}\int_{\partial\mathcal{O}} D(\varphi)n\,\mathrm{d}\gamma + \left(2\mu J^{-1}\int_{\partial\mathcal{O}} y \times D(\varphi)n\; d\gamma\right) \times y & \text{in } \mathcal{O} \end{cases}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z(t,\cdot), \varphi\rangle_{L^2(\mathbb{R}^3)} = \langle z(t,\cdot), \mathcal{A}_2\varphi\rangle_{L^2(\mathbb{R}^3)}.$$

The fluid structure operator (monolithic approach)

For $q > 1$ set: $\mathbb{H}_q = \left\{ \varphi \in \left[L^q(\mathbb{R}^3)\right]^3 \mid \operatorname{div} \varphi = 0 \text{ in } \mathbb{R}^3, \quad D(\varphi) = 0 \text{ in } \mathcal{O} \right\}$;

$\mathbb{P}_q$ is the projector from $L^q(\mathbb{R}^3)$ onto $\mathbb{H}_q$ and $\mathbb{A}_q : \mathcal{D}(\mathbb{A}_q) \to \mathbb{H}_q$ is defined by

$$\mathcal{D}(\mathbb{A}_q) = \left\{ \varphi \in \left[W^{1,q}(\mathbb{R}^3)\right]^3 \;\middle|\; \begin{array}{c} \varphi_{|\mathbb{R}^3 \setminus \mathcal{O}} \in \left[W^{2,q}(\Omega \setminus \mathcal{O})\right]^3 \\ \operatorname{div} \varphi = 0 \text{ in } \mathbb{R}^3 \\ \nabla\varphi + (\nabla\varphi)^* = 0 \text{ in } \mathcal{O} \end{array} \right\}.$$

$$\mathbb{A}_q \varphi = \mathbb{P}_q \mathcal{A}_q \varphi \qquad\qquad (\varphi \in \mathcal{D}(\mathbb{A}_q)),$$

with $D(\varphi) = \frac{1}{2}\left(\nabla\varphi + (\nabla\varphi)^*\right)$, $\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(\mathbb{A}_q)$ and

$$\mathcal{A}_q \varphi = \begin{cases} \mu \Delta \varphi & \text{in } \mathbb{R}^3 \setminus \mathcal{O}, \\ 2\mu m^{-1} \int_{\partial\mathcal{O}} D(\varphi) n \, \mathrm{d}\gamma + \left(2\mu J^{-1} \int_{\partial\mathcal{O}} y \times D(\varphi) n \, d\gamma\right) \times y & \text{in } \mathcal{O} \end{cases}$$

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Existing results on the fluid structure operator

- $\mathbb{A}_2$ generates an anlytic semigroup in $\mathbb{H}_2$, see [4].
- In [5] it has been shown that the part of $\mathbb{A}_q$ in an appropriate subspace of $\mathbb{H}_q$ generates an analytic semigroup.
- For a bounded set $\Omega \subset \mathbb{R}^3$, the operator $\mathbb{A}_q$ on $\mathbb{H}_q(\Omega)$ has the infinite time maximal regularity property, see [2], [3]. For $\Omega = \mathbb{R}^3$ we have finite time maximal regularity.

References

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[2] D. Maity and M. Tucsnak, in *Particles in Flows*, Springer, 2017.

[3] D. Maity and M. Tucsnak, in vol. 710 of *Contemp. Math.*, 2018.

[4] T. Takahashi and M. Tucsnak, *J. Math. Fluid Mech.* **6** (2004), pp. 53–77.

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A resolvent estimate

Theorem.(Ervedoza, Maity and M.T., Math. Annalen, 2023)
Let $q > 1$, $\theta \in (\pi/2, \pi)$ and $\mathcal{S}_\theta = \{\lambda \in \mathbb{C} \setminus \{0\}, |\mathrm{Arg}\,(\lambda)| \leq \theta\}$. The there exists $M > 0$ with $\|\lambda(\lambda I - \mathbb{A}_q)^{-1}\| \leqslant M$ for $\lambda \in \mathcal{S}_\theta$.

Main steps of the proof (strongly inspired by [1]):

- For $q \in (1, 3/2)$ combine the *fluid-structure estimate* for bounded domains with the classical estimate for Stokes operator in the whole space.
- Use interpolation for $q \in [3/2, 2)$ and duality for $q > 2$.

References

[1] W. Borchers and H. Sohr, *On the semigroup of the Stokes operator for exterior domains in L^q-spaces*, Math. Z., 196 (1987), pp. 415–425.

[2] H.Iwashita, *Mathematische Annalen* **285** (1989), 265-288.

Main ingredient: decay estimates for the linearized system

Theorem. Let $\mathbb{T}^q$ be the analytic semigroup on $\mathbb{H}_q$ generated by $\mathbb{A}_q$. Then:

(i) Let $1 < q < \infty$. Let $R_0 > 0$ be such that $\overline{\mathcal{O}} \subset B_{R_0}$. Then for any $R > R_0$, there exists a constant $C > 0$, depending on q and R, such that

$$\|\mathbb{T}_t^q U\|_{q,B_R} \leqslant C t^{-\frac{3}{2q}} \|U\|_{\mathbb{H}_q} \qquad (t > 1,\ U \in \mathbb{H}_q).$$

(ii) Let $1 < q \leq r < \infty$ and $\sigma = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$. Then there exists a constant $C > 0$, depending on q and r, such that

$$\|\mathbb{T}_t^q U\|_{\mathbb{H}_r} \leqslant C t^{-\sigma} \|U\|_{\mathbb{H}_q} \qquad (t > 0,\ U \in \mathbb{H}_q).$$

(iii) Let $1 < q \leqslant r \leqslant 3$. Then there exists a constant $C > 0$, depending on q and r, such that

$$\|\nabla \mathbb{T}_t^q U\|_{r,E} \leqslant C t^{-\sigma - 1/2} \|U\|_{\mathbb{H}_q} \qquad (t > 0,\ U \in \mathbb{H}_q).$$

The fixed point procedure

We follow a procedure going back to Kato by introducing the space

$$\mathcal{C} = \left\{ V = \begin{bmatrix} v \\ \ell \\ \omega \end{bmatrix} \text{ with } t^{1/4}V \in C_b^0([0,\infty);\mathbb{X}^6),\ t^{1/2}V \in C_b^0([0,\infty);\mathbb{X}^\infty) \right.$$

$$\left. \text{and } \min\{t^{1/2},1\}\nabla v \in C^0([0,T];\left[L^3(E)\right]^9) \right\},$$

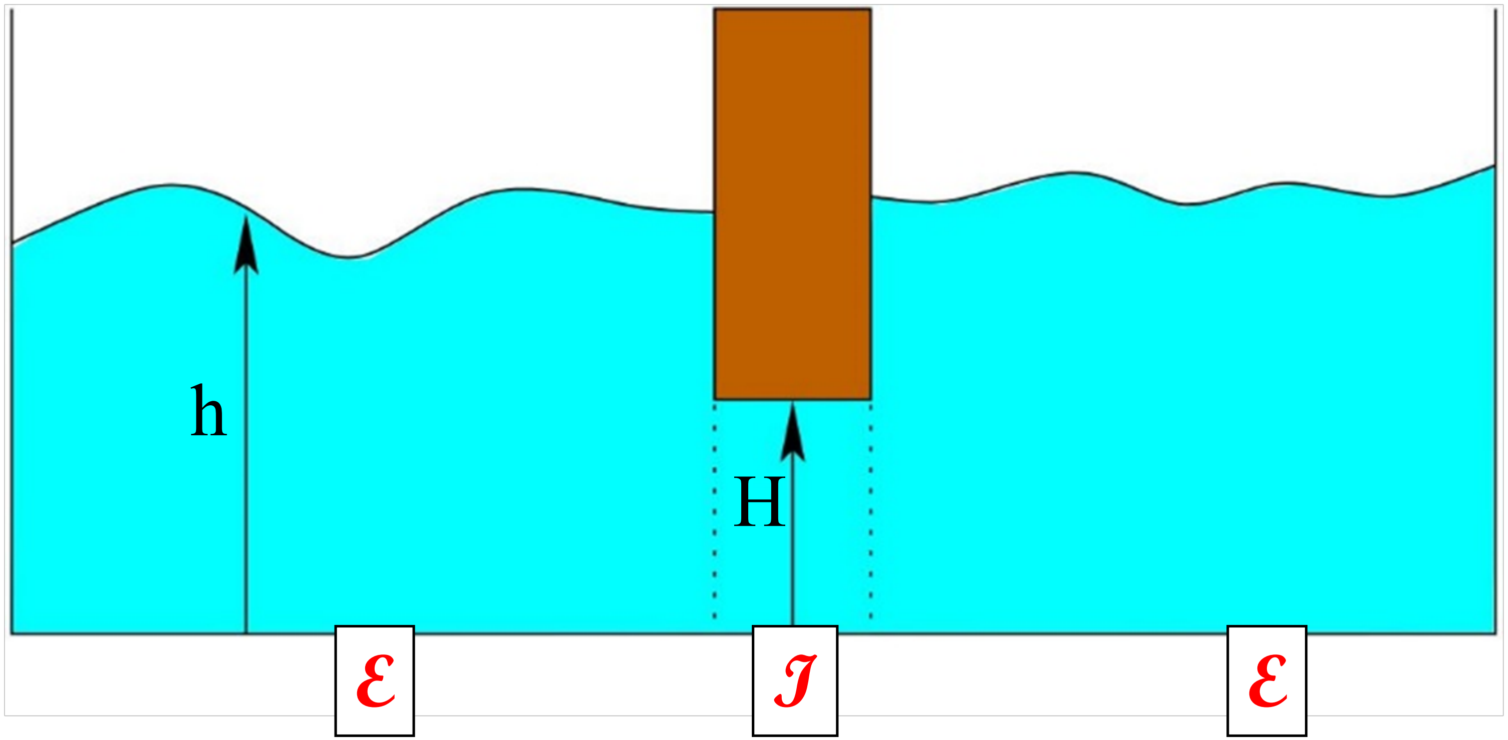
and the sequence (V_n) defined by

$$V(t) = \mathbb{T}_t V_0 + \int_0^t \mathbb{T}_{t-s}\,\mathbb{P}\,\mathrm{div}\,F((V_n(s))\,\mathrm{d}s \qquad (t \geqslant 0),$$

and we show that $V_n \to V$ in $\mathcal{C}$.

The point absorber problem

A model for a point absorber (WEC)



$$q(t,x) = h(t,x)v(t,x)$$

The governing equations
(Maity, San Martin, Takahashi and MT, 2019)

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0 && (t>0,\ x\in \mathcal{I}\cup\mathcal{E}),\\
\frac{\partial}{\partial t}\left(\frac{q}{H}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) &= 0 && (t>0,\ x\in \mathcal{I}),\\
\frac{\partial}{\partial t}\left(\frac{q}{h}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) &= \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial q}{\partial x}\right) && (t>0,\ x\in \mathcal{E}),\\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,a^+) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,a^-) && (t>0),\\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,b^-) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,b^+) && (t>0),\\
M\ddot{H}(t) &= -Mg + \int_a^b p(t,x)\,\mathrm{d}x && (t>0).
\end{aligned}
$$

Local existence (Maity, San Martin, Takahashi and MT, 2019)

Theorem. (working also for infinite containers)

Assume that $|H_0|+\|h_0\|_{H^1(\mathcal{E})}+\|q_0\|_{H^1(0,\ell)} \leqslant K, \quad \frac{1}{K} \leqslant h_0(x) \leqslant K$ for $x \in \mathcal{E}$.
Then, there exists $T = T(K) > 0$ such that we have unique strong solution

$$H \in H^2(0,T), \quad h \in H^1(0,T;H^1(\mathcal{E})) \cap C^1([0,T];L^2(\mathcal{E})),$$

$$q \in C^0([0,T];H^1(0,\ell))$$

$$q_{|\mathcal{E}} \in H^1(0,T;L^2(\mathcal{E})) \cap C^0([0,T];H^1(\mathcal{E})) \cap L^2(0,T;H^2(\mathcal{E})),$$

$$q_{|\mathcal{I}} \in H^1(0,T;\mathcal{P}_1(\mathcal{I})),$$

$$p_{|\mathcal{I}} \in L^2(0,T;\mathcal{P}_2(\mathcal{I})).$$

$$\frac{1}{K_T} \leqslant h_0(t,x), H(t) \leqslant K_T, \qquad \text{for all } t \in (0,T), x \in \mathcal{E},$$

$$H(t) < \min(h(t,a^-), h(t,b^+)) \qquad \text{for all } t \in (0,T).$$

Global existence (Maity, San Martin, Takahashi and MT, 2019 + in progress)

For initial data close to equilibrium and bounded container we have

$$H \in \overline{H} + H^2(0,\infty), \quad h \in \overline{h} + H^1(0,\infty; H^1(\mathcal{E})) \cap C^1_b([0,\infty); L^2(\mathcal{E})),$$

$$q \in C_b([0,\infty); H^1(0,\ell))$$

$$q_{|\mathcal{E}} \in H^1(0,\infty; L^2(\mathcal{E})) \cap C_b([0,\infty); H^1(\mathcal{E})) \cap L^2(0,\infty; H^2(\mathcal{E})),$$

$$q_{|\mathcal{I}} \in H^1(0,\infty; \mathcal{P}_1(\mathcal{I}))$$

$$p_{|\mathcal{I}} \in \frac{Mg}{|\mathcal{I}|} + L^2(0,\infty; \mathcal{P}_2(\mathcal{I}))$$

$$\int_{\mathcal{E}} h(t,x)\,\mathrm{d}x + H(t)(b-a) = M\frac{|\mathcal{E}|}{|\mathcal{I}|} + \overline{H}\ell \quad (t \geq 0),$$

$$h(t,x) \geqslant \frac{\overline{h}}{2}, \qquad H(t) \geqslant \frac{\overline{H}}{2}, \text{ for all } t \in (0,\infty), x \in \mathcal{E},$$

$$H(t) < \min(h(t,a^-), h(t,b^+)) \qquad \text{for all } t \in (0,\infty).$$

Open questions for floating bodies

- Unbounded fluid (work in progress)
- Non vertical walls
- 2D Saint-Venant (even for vertical walls)
- Solid moving also horizontally
- Other fluid models like Euler (see Lannes), Navier-Stokes,…
- …