Lecture 3
Water waves and floating bodies.
Deep water models

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Motivation

In very broad terms: modelling floating structures such as: Wave Energy Converters (WECs), floating wind turbines, ...

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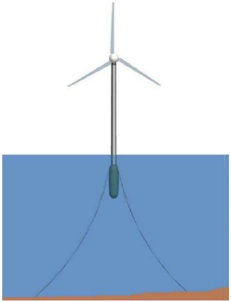
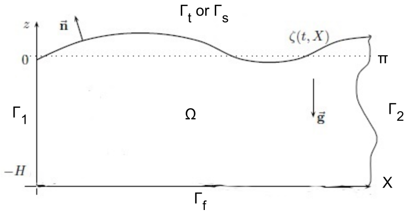


Floating wind turbine

Basic assumptions

- Fluid: inviscid, incompressible, irrotational, constant density;
- Domain: 2-dimensional; bounded by free surface, fixed bottom (flat).



Notation

x: horizontal variable, z: vertical variable
$\eta(t,x)$: The elevation of the free surface;
$V(t,x,z)$: The velocity of the fluid particle (x,z) at time t;
$\varphi(t,x,z)$: The velocity potential;
Γ_f: The fixed bottom $\Gamma_f = \{(x,-a) | x \in \mathbb{R}\}$;
Γ_s: The undistured free surface $\Gamma_s = \{(x,0) | x \in \mathbb{R}\}$;
Γ_t: Current free surface $\Gamma_t = \{(x,\eta(t,x)) | x \in \mathbb{R}\}$.

▶1.1 Governing equations

- Incompressible: $\nabla \cdot V = 0 \quad$ in Ω
- Irrotational: $V = \nabla\varphi \quad$ in Ω
- Bernoulli's equation:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0 \quad \text{on} \quad \Gamma_t$$

- Kinematic condition:
  The fluid particle keep staying on the free surface.

$$V \cdot \vec{n}\sqrt{1 + |\nabla\eta|^2} = \eta_t \quad \text{on } \Gamma_t$$

where $\vec{n}$ is unit outer normal.

- Impermeable boundary Γ_f:

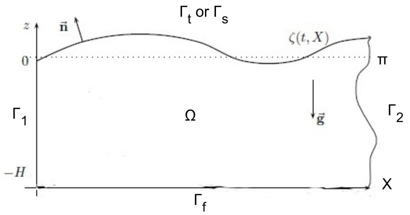
$$V \cdot \vec{n} = \partial_{\vec{n}}\varphi = 0 \quad \text{on } \Gamma_f$$

So one obtains the Zakharov/Craig-Sulem formulation:

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega \\ \varphi_t + \dfrac{1}{2}|\nabla\varphi|^2 + g\eta = 0 & \text{on } \Gamma_t \\ \sqrt{1+|\nabla\eta|^2}\,\partial_{\vec{n}}\varphi = \eta_t & \text{on } \Gamma_t \\ \partial_{\vec{n}}\varphi = 0 & \text{on } \Gamma_f \end{cases}$$

Linearized system

$$\begin{cases} \Delta\varphi(t,x,z) = 0 & (x,\ y \in \mathbb{R}), \\ \varphi_t(t,x,0) + g\eta(t,x) = 0 & (x \in \mathbb{R}), \\ \partial_z\varphi(t,x,0) = \eta_t(t,x) & (x \in \mathbb{R}), \\ \partial_z\varphi(t,x,-H) = 0 & (x \in \mathbb{R}). \end{cases}$$



The Dirichlet and Dirichlet to Neumann maps

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \psi = v & \text{on } \Gamma_s \\ \partial_{\vec{n}}\psi = 0 & \text{on } \Gamma_f \end{cases}$$

Let D denote "Dirichlet map" defined by

$$D : v \longmapsto Dv = \psi.$$

Let Λ denote "Dirichlet-to-Neumann map" defined by

$$\Lambda : v \longmapsto \frac{\partial(Dv)}{\partial\nu}|_{\Gamma_s}.$$

Operator form of the linearized system

Let $\psi = \psi(t; x, z) := \frac{\partial \varphi}{\partial t}(t; x, z)$

$$\begin{cases} \Delta\psi(t,x,z) = 0 & (x, z \in \mathbb{R}), \\ \psi(t,x,0) + g\eta(t,x) = 0 & (x \in \mathbb{R}), \\ \partial_z\psi(t,x,0) = \eta_{tt}(t,x) & (x \in \mathbb{R}), \\ \partial_z\psi(t,x,-H) = 0 & (x \in \mathbb{R}). \end{cases}$$

Let $w(t,x) = \psi(t,x,0)$. Then

$$\begin{cases} w_{tt}(t,x) + g(\Lambda w)(t,x) = 0 & (t > 0,\ x \in \mathbb{R}), \\ w(0,x) = w_0(x) & (x \in \mathbb{R}). \end{cases}$$

Self-adjointness of the Dirichlet-to-Neumann operator on $L^2(\mathbb{R})$.

Theorem 1

The Dirichlet to Neumann map Λ is a self-adjoint operator on $L^2(\mathbb{R})$, with domain $H^1(\mathbb{R})$.

Proof.

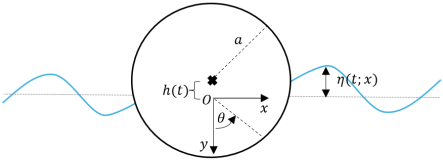
Poisson's formula yields $(Dv)(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\eta)^2+y^2} v(\eta)\,\mathrm{d}\eta$,

$$\frac{\partial(Dv)}{\partial y}(x,0) = \frac{1}{\pi}\mathrm{vp}\int_{\mathbb{R}} \frac{v(\eta)}{(x-\eta)^2}\,\mathrm{d}\eta = \frac{1}{\pi}\mathrm{vp}\int_{\mathbb{R}} \frac{v'(\eta)}{x-\eta}\,\mathrm{d}\eta.$$

Thus Λv is the *Hilbert transform* of v'. □

A toy floating body model

- Disk with density ρ, radius a and mass center $h(t)$.
- Forces acting on the disk: hydrostatic restoring force and the resultant force from the hydrodynamic pressure.
- Physical advantages



Let $\psi = \psi(t; x, y) := \frac{\partial \varphi}{\partial t}(t; x, y)$. The system that couples the motion of the disk and the water waves is given by the following:

Coupled equations for h and ψ

$$\frac{1}{2}\pi\rho a^2 \ddot{h}(t) = -2\rho g a h(t) + 2\rho a \int_0^{\pi/2} \psi(t; a\sin\theta, a\cos\theta)\cos\theta \, \mathrm{d}\theta$$

$$\frac{\partial^2 \psi}{\partial x^2}(t; x, y) + \frac{\partial^2 \psi}{\partial y^2}(t; x, y) = 0 \qquad (x^2 + y^2 > a^2, y > 0)$$

$$\frac{\partial^2 \psi}{\partial t^2}(t; x, 0) - g\frac{\partial \psi}{\partial y}(t; x, 0) = 0 \qquad (|x| > a)$$

$$\frac{\partial \psi}{\partial r}(t; a\sin\theta, a\cos\theta) = \ddot{h}(t)\cos\theta \qquad \left(-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}\right),$$

Potential functions

$$\psi(t;x,y) = \ddot{h}(t)\psi_1(x,y) + (Dw(t,\cdot))(x,y), \qquad (1)$$

where ψ_1 and Dv are defined as solutions of

$$\begin{aligned}
\frac{\partial^2\psi_1}{\partial x^2}(x,y) + \frac{\partial^2\psi_1}{\partial y^2}(x,y) = 0 \qquad & (x^2+y^2>a^2, y>0),\\
\frac{\partial\psi_1}{\partial r}(a\sin\theta, a\cos\theta) = \cos\theta \qquad & (-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}),\\
\psi_1(x,0) = 0 \qquad & (|x|>a),
\end{aligned}$$

respectively of

$$\begin{aligned}
\frac{\partial^2(Dv)}{\partial x^2}(x,y) + \frac{\partial^2(Dv)}{\partial y^2}(x,y) = 0 \qquad & (x^2+y^2>a^2, y>0),\\
\frac{\partial(Dv)}{\partial r}(a\sin\theta, a\cos\theta) = 0 \qquad & (-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}),\\
(Dv)(x,0) = v(x) \qquad & (|x|>a).
\end{aligned}$$

The governing equations (I)

Setting $(\Lambda w(t,\cdot))(x) = -\frac{\partial(Dw(t,\cdot))}{\partial y}(x,0)$, with $t>0, |x|>a$,

$$\frac{\partial^2 w}{\partial t^2}(t;x) - g\ddot{h}(t)\frac{\partial \psi_1}{\partial y}(x,0) + g(\Lambda w)(t;x) = 0,$$

$$\frac{1}{2}\pi\rho a^2\ddot{h}(t) = -\,2\rho g a h(t) + 2\rho a\ddot{h}(t)\int_0^{\pi/2}\psi_1(a\sin\theta, a\cos\theta)\cos\theta\,\mathrm{d}\theta$$
$$+\,2\rho a\int_0^{\pi/2}(Dw)(t;a\sin\theta, a\cos\theta)\cos\theta\,\mathrm{d}\theta.$$

The governing equations (II)

$$M(x) = \begin{bmatrix} g\frac{\partial\psi_1}{\partial y}(x,0) & -1 \\ \frac{1}{2}\pi\rho a^2 - 2\rho a \int_0^{\pi/2} \psi_1(a\sin\theta, a\cos\theta)\cos\theta\,\mathrm{d}\theta & 0 \end{bmatrix},$$

so that

$$M(x)\begin{pmatrix} \ddot{h}(t) \\ \ddot{w}(t) \end{pmatrix} = \begin{pmatrix} g(\Lambda w)(t;x) \\ -2\rho g a h(t) + 2\rho a \int_0^{\pi/2} (Dw)(t; a\sin\theta, a\cos\theta)\cos\theta\,\mathrm{d}\theta \end{pmatrix}.$$

Operator form of the governing equations

We have obtained the differential equation

$$\begin{pmatrix} \ddot{h} \\ \ddot{w} \end{pmatrix} + A_0 \begin{pmatrix} h \\ w \end{pmatrix} = 0, \tag{2}$$

where $w(t;x) = \psi(t;x,0)$ for $|x| > a$, and A_0 is a block operator that, when applied to (h, w), encapsulates:

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- The mass of the body
- The added mass effect produced by the waves
- Boundary conditions (including Dirichlet to Neumann map)

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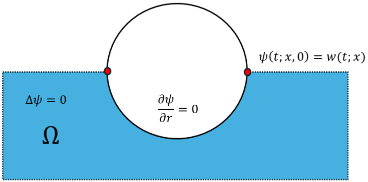
Mathematical challenges:

Proving that Λ and A_0 are self-adjoint

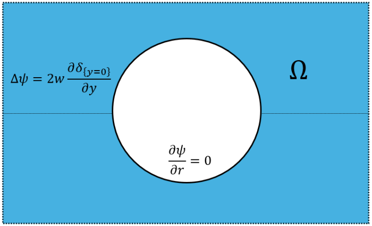
Solution

Reflection argument to work on the whole plane except the disk.

A possible approach

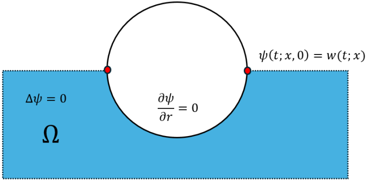


Original domain



Extended domain

A possible approach



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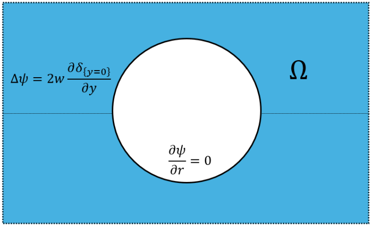

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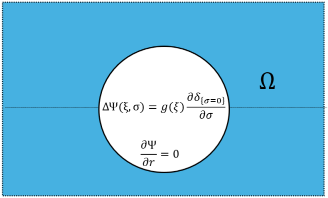

Work inside the disk

$$\Psi(\xi,\sigma) = (G * g\frac{\partial\delta_{\sigma=0}}{\partial\sigma})(\xi,\sigma)$$

Thanks for your attention !