Lecture 4
Water waves and floating bodies.
Shallow water modelling

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Motivation

In very broad terms: modelling floating structures such as: Wave Energy Converters (WECs), floating wind turbines, ... when they are close to the shore.

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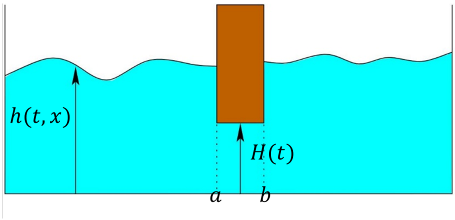
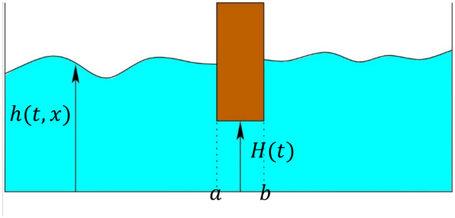


Floating wind turbine

Notation



Floating wind turbine

Unknown functions : h, H and q, Q with $q = vh$.

The governing equations (fluid alone)

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0 && (t > 0,\ x \in [0, \ell]), \\
\frac{\partial}{\partial t}\left(\frac{q}{h}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) &= \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial q}{\partial x}\right) && (t > 0,\ x \in [0, \ell]), \\
q(t, 0) = q(t, L) &= 0.
\end{aligned}
$$

Lagrangian structure (I)

K_f is the kinetic energy and U_f is the potential energy of the fluid:

$$K_f = \frac{1}{2}\int_{(0,\ell)} \rho h v^2 \,\mathrm{d}x, \qquad U_f = \frac{1}{2}\int_{(0,\ell)} \rho g h^2 \,\mathrm{d}x.$$

The associated Lagrangian is

$$\mathcal{L}_f(h,q,H,\lambda_1) = \int_0^T \left\{ \int_0^\ell \left[\frac{1}{2}\rho\frac{q^2}{h} - \frac{1}{2}\rho g h^2 + \rho\lambda_1\left(\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x}\right)\right] \mathrm{d}x, \right.$$

where $\lambda_1(t,x)$ is a *Lagrange multiplier*.

Lagrangian structure (II)

The governing equations are obtained by imposing that $\delta\mathcal{L}_f = 0$ for any virtual displacement $\delta h(t,x)$ and $\delta\varphi(t,x)$ such that

$$\delta h(0,x) = \delta h(T,x) = \delta H(0) = \delta H(T) = \delta\varphi(0,x) = \delta\varphi(T,x) = 0,$$

$$\delta\varphi(t,0) = \delta\varphi(t,\ell) = 0,$$

where

$$\frac{\partial\varphi}{\partial t} = q \qquad \text{and} \qquad \frac{\partial(\delta\varphi)}{\partial t} = \delta q.$$

The Lagrangian formalism

$$\mathcal{L}(h,q,H,\lambda_1,\lambda_2) = \int_0^T \left\{ \int_0^\ell \left[\frac{1}{2}\rho\frac{q^2}{h} - \frac{1}{2}\rho g h^2 + \rho\lambda_1\left(\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x}\right)\right] \mathrm{d}x \right.$$

$$\left. + \int_{\mathcal{I}} \left[\lambda_2\Big(H - h\Big)\right] \mathrm{d}x + \frac{1}{2}M\dot{H}^2 - MgH \right\} \mathrm{d}t.$$

Imposing that $\delta\mathcal{L} = 0$ for any virtual displacement $\delta h(t,x)$, $\delta H(t)$ and $\delta\varphi(t,x)$ such that

$$\delta h(0,x) = \delta h(T,x) = \delta H(0) = \delta H(T) = \delta\varphi(0,x) = \delta\varphi(T,x) = 0,$$

$$\delta\varphi(t,0) = \delta\varphi(t,\ell) = 0$$

we obtain

The coupled governing equations

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0 && (t>0,\ x \in \mathcal{I}\cup\mathcal{E}),\\
\frac{\partial}{\partial t}\left(\frac{q}{H}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) &= 0 && (t>0,\ x \in \mathcal{I}),\\
\frac{\partial}{\partial t}\left(\frac{q}{h}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) &= \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial q}{\partial x}\right) && (t>0,\ x \in \mathcal{E}),\\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,a^+) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,a^-),\\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,b^-) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,b^+),\\
M\ddot{H}(t) &= -Mg + \int_a^b p(t,x)\,\mathrm{d}x && (t>0).
\end{aligned}
$$

Stationary states

One can check that the stationary solutions of (**??**)–(**??**) can be parametrized by $\overline{H} > 0$ by setting

$$\overline{h} := \overline{H} + \frac{M}{b-a}, \quad \overline{p} := \frac{Mg}{b-a}$$

and in that case

$$h^S(x) = \begin{cases} \overline{h} & x \in \mathcal{E} \\ \overline{H} & x \in \mathcal{I} \end{cases}, \qquad q^S(x) = 0 \quad \text{and} \quad p^S(x) = \begin{cases} 0 & x \in \mathcal{E} \\ \overline{p} & x \in \mathcal{I} \end{cases}.$$

Main result

Assume that the initial data are close to an equilibrium with $H = \overline{H} > 0$, Then we have solutions with

$$H \in \overline{H} + H^2(0,\infty), \quad h \in \overline{h} + H^1(0,\infty; H^1(\mathcal{E})) \cap C_b^1([0,\infty); L^2(\mathcal{E})),$$

$$q \in C_b([0,\infty); H^1(0,\ell))$$

$$q_{|\mathcal{E}} \in H^1(0,\infty; L^2(\mathcal{E})) \cap C_b([0,\infty); H^1(\mathcal{E})) \cap L^2(0,\infty; H^2(\mathcal{E})),$$

$$q_{|\mathcal{I}} \in H^1(0,\infty; \mathcal{P}_1(\mathcal{I}))$$

$$p_{|\mathcal{I}} \in \frac{Mg}{|\mathcal{I}|} + L^2(0,\infty; \mathcal{P}_2(\mathcal{I}))$$

$$\int_{\mathcal{E}} h(t,x)\,\mathrm{d}x + H(t)(b-a) = M\frac{|\mathcal{E}|}{|\mathcal{I}|} + \overline{H}\ell \quad (t \geq 0).$$

Equivalent form of the governing equation

$$\dot{H} = -\frac{q_b - q_a}{b-a} \qquad (t \geqslant 0),$$

$$\frac{\partial h}{\partial t}(t,x) + \frac{\partial q}{\partial x}(t,x) = 0 \qquad (t > 0,\ x \in \mathcal{E}),$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{gh^2}{2}\right) = h\frac{\partial}{\partial x}\left(\frac{\mu}{h}\frac{\partial q}{\partial x}\right) \qquad (t > 0,\ x \in \mathcal{E}),$$

$$q(t,0) = q(t,\ell) = 0 \qquad (t > 0),$$

$$q(t,a) = q_a(t), \quad q(t,b) = q_b(t) \qquad (t > 0),$$

$$\begin{bmatrix} \dot{q}_a \\ \dot{q}_b \end{bmatrix} = R\left(H, q(\cdot,a), \frac{\partial}{\partial x}q(\cdot,a^-), h(\cdot,a^-), q(\cdot,b), \frac{\partial}{\partial x}q(\cdot,b^+), h(\cdot,b^+)\right),$$

with $q_a(t) := q(t,a^-) = q(t,a^+), \quad q_b(t) := q(t,b^-) = q(t,b^+).$

Linearization

$$\dot{\widetilde{H}} = -\frac{q_b - q_a}{b-a} \qquad (t \geqslant 0),$$

$$\frac{\partial \widetilde{h}}{\partial t}(t,x) + \frac{\partial q}{\partial x}(t,x) = 0 \qquad (t > 0,\ x \in \mathcal{E}),$$

$$\frac{\partial q}{\partial t} + g\overline{h}\frac{\partial \widetilde{h}}{\partial x} - \mu \frac{\partial}{\partial x}\left(\frac{\partial q}{\partial x}\right) = f_1 \qquad (t > 0,\ x \in \mathcal{E}),$$

$$q(t,0) = q(t,\ell) = 0 \qquad (t > 0),$$

$$q(t,a) = q_a(t), \quad q(t,b) = q_b(t) \qquad (t > 0),$$

$$\begin{bmatrix} \dot{q}_a \\ \dot{q}_b \end{bmatrix} = S_0(\overline{H}) \begin{bmatrix} \frac{\mu}{\overline{H}} \frac{q(\cdot,b)-q(\cdot,a)}{b-a} + g\left(\widetilde{h}(\cdot,a^-) - \widetilde{H}\right) - \frac{\mu}{\overline{h}}\frac{\partial q}{\partial x}(\cdot,a^-) \\ -\frac{\mu}{\overline{H}} \frac{q(\cdot,b)-q(\cdot,a)}{b-a} - g\left(\widetilde{h}(\cdot,b^+) - \widetilde{H}\right) + \frac{\mu}{\overline{h}}\frac{\partial q}{\partial x}(\cdot,b^+) \end{bmatrix} + f_2$$

Wellposedness of the linearized problem

Assume that $e^{\eta(\cdot)} f_1 \in L^2(0,\infty; L^2(\mathcal{E}))$, $e^{\eta(\cdot)} f_2 \in L^2(0,\infty;\mathbb{R}^2)$. Then the linearized system admits a unique solution with

$$e^{\eta(\cdot)} \widetilde{H} \in H^1(0,\infty), \quad e^{\eta(\cdot)} \widetilde{h} \in H^1(0,\infty; H^1(\mathcal{E})),$$

$$e^{\eta(\cdot)} q \in H^1(0,\infty; L^2(\mathcal{E})) \cap C_b([0,\infty); H^1(\mathcal{E})) \cap L^2(0,\infty; H^2(\mathcal{E})),$$

$$e^{\eta(\cdot)} q_a \in H^1(0,\infty), \quad e^{\eta(\cdot)} q_b \in H^1(0,\infty),$$

$$\int_{\mathcal{E}} \widetilde{h}(t,x)\,\mathrm{d}x + \widetilde{H}(t)(b-a) = 0 \quad (t \geq 0).$$

Moreover, there exists $C > 0$ such that

$$\|e^{\eta(\cdot)} \widetilde{H}\|_{H^1(0,\infty)} + \|e^{\eta(\cdot)} \widetilde{h}\|_{H^1(0,\infty;H^1(\mathcal{E}))} + \|e^{\eta(\cdot)} q\|_{H^1(0,\infty;L^2(\mathcal{E}))\cap C_b([0,\infty);H}$$

$$+ \|e^{\eta(\cdot)}(q_a, q_b)\|_{H^1(0,\infty;\mathbb{R}^2)}$$

$$\leq C\left(|\widetilde{H}_0| + \|\widetilde{h}_0\|_{H^1(\mathcal{E})} + \|q_0\|_{H^1(\mathcal{E})} + \left\| e^{\eta(\cdot)}(f_1, f_2) \right\|_{L^2(0,\infty;L^2(\mathcal{E})\times\mathbb{R}^2)}\right).$$

Fixed point procedure

$$\mathfrak{B}_\varepsilon := \left\{ (f_1, f_2) \in L^2(0,\infty; L^2(\mathcal{E}) \times \mathbb{R}^2) \; ; \; \left\| e^{\eta(\cdot)} (f_1, f_2) \right\|_{L^2(0,\infty; L^2(\mathcal{E}) \times \mathbb{R}^2)} \leq \varepsilon \right\}$$

and the map

$$\Xi : (f_1, f_2) \in \mathfrak{B}_\varepsilon \mapsto (F_1(\widetilde{h}, q), F_2(\widetilde{H}, \widetilde{h}, q, q_b, q_a))$$

where $(\widetilde{H}, \widetilde{h}, q, q_b, q_a)$ is the solution of the linearized problem.
We end by proving that for ε small enough Ξ is a strict contraction of $\mathfrak{B}_\varepsilon$.

Some open questions

- Non vertical walls for the floating body
- 2D Saint-Venant (even for vertical walls)
- Solid moving also horizontally
- Control issues (see Ringwood's book)
- Other fluid models like Euler (see Lannes), Navier-Stokes,

Thanks for your attention !