Loewner Framework for data-driven reduced order modeling (with pH-structure preserving)

... a bridge between realization, approximation and identification

C. Poussot-Vassal
October 18, 2025
coll. with P. Vuillemin, D. Matignon, G. Haine & M. Fournié

Forewords

Advancement (energy driven approach...)

Lecture Motivations, pH energy philosophy (Denis Matignon)

Lecture pH in 1-D, infinite dimension (Andrea Brugnoli)

Lecture pH in n-D, infinite dimension (Denis Matignon)

Lecture PFEM for pH (Michel Fournié)

Lecture PFEM for pH with applications (Michel Fournié)

Lecture **Model reduction, approximation and passivity in the LF**

Lab Discretize a PDE with PFEM (& `SCRIMP`)

Lab **Simplify the PFEM using LF with passivity preservation (& `+lf`)**

Lecture Open perspectives (Andrea Brugnoli)

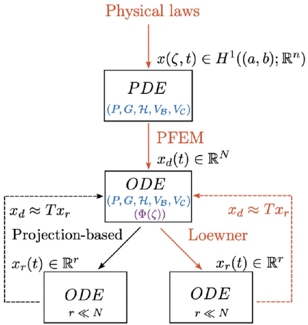


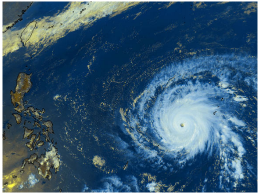
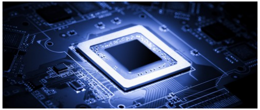
Figure by J.P. Toledo-Zucco

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

A.C. Antoulas, *"A new result on passivity preserving model reduction"*, Systems & Control Letters, vol. 54, 2005.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Dynamical models are centrals tools in engineering...

Digitalisation and computer-based modeling for

- simulation, optimisation, understanding
- control, estimation, analysis...

However
Finite machine precision, computational burden, memory management and actual solvers

- induces important time consumption
- generate inaccurate results
- limit the class of models to deal with
- limit the amount of data to treat

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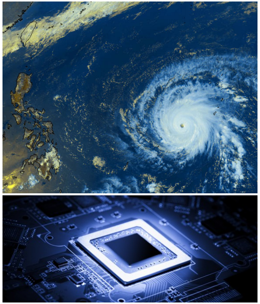

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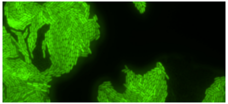
Develop robust and efficient tools to construct simplified dynamical models

Forewords

Dynamical models, what for?

… and in systems and control engineering

- **for verification and validation**
  (μ, $\mathcal{H}_\infty$-norm, pseudo-spectra, Monte Carlo)
- **for detection**
  (fault isolation, param. estim.)
- **for uncertainty propagation**
  (Multi Disc. Optim., robust optim.)
- **for feedback control synthesis**
  ($\mathcal{H}_\infty/\mathcal{H}_2$-norm, MPC, adaptive)



Complex models

- important sim. time
- memory burden
- inaccurate results
- limit model class

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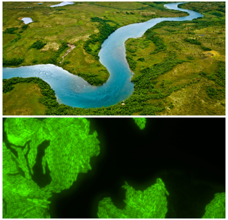

Complex models

- important sim. time
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$\Longrightarrow$

Model simplification

Simplified models

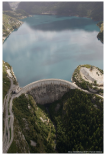
- reduced sim. time
- memory saving
- accurate results
- rational model

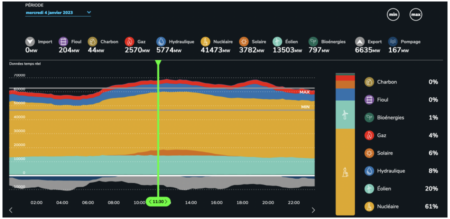
Forewords

Use-case: EDF hydroelectric open-channel (model-driven)

Hydraulic-driven electricity

- Dams & run-of-the-river ($\approx 10\%$)
- Run-of-the-river ($\approx 5\%$)
- Rely on open-channel hydraulic systems
- Need for analysis and control





January 4th, 2023

Forewords

Use-case: EDF hydroelectric open-channel (model-driven)

Time-domain

$$
\begin{aligned}
\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/S)}{\partial x} + gS\frac{\partial H}{\partial x} &= gS(I-J),
\end{aligned}
$$

Frequency-domain

$$
\begin{aligned}
h(s,x) &= G_i(s,x)q_i(s) - G_o(s,x)q_o(s) \\
G_i(s,x) &= \frac{\lambda_1(s)e^{\lambda_2(s)L+\lambda_1(s)x} - \lambda_2(s)e^{\lambda_1(s)L+\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\
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Both are x-position dependent, hard to simulate in practice...

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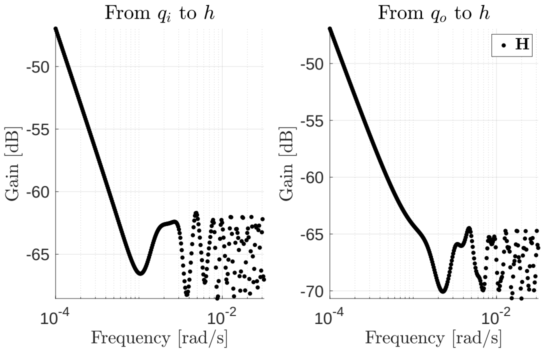

- Model
  (integral and delay action)
- $E\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$
  $\mathbf{H}_{51}$
  (full interpolant, unstable)
- $\hat{E}\dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{B}\mathbf{u}$
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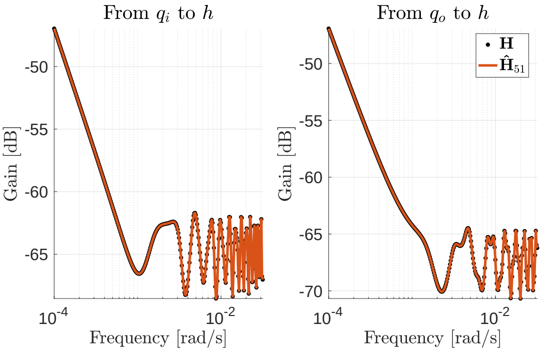

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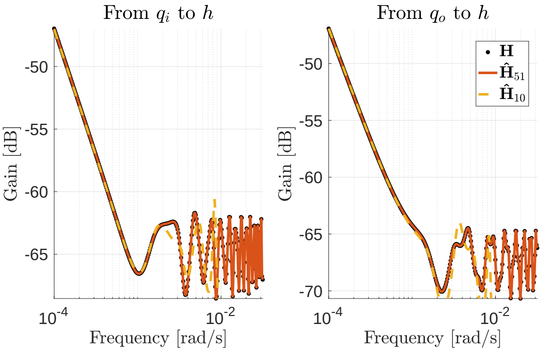

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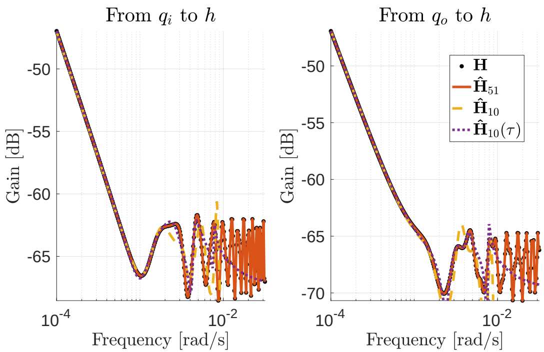

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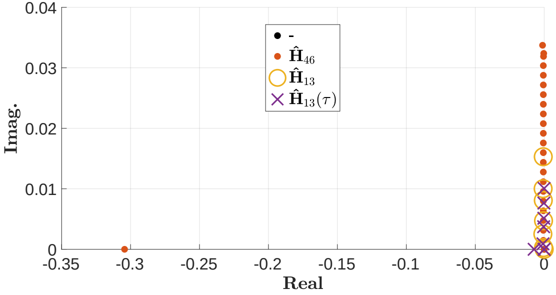

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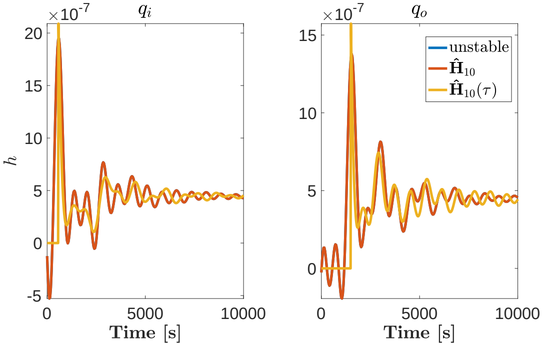

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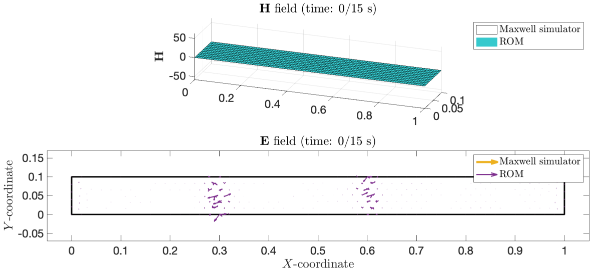
ANTENNA RESEARCH

Antenna models

- **to optimize parameters**
- **for polar computation**

Blend physics from

- **Maxwell equations**
- **Kirchoff equations**

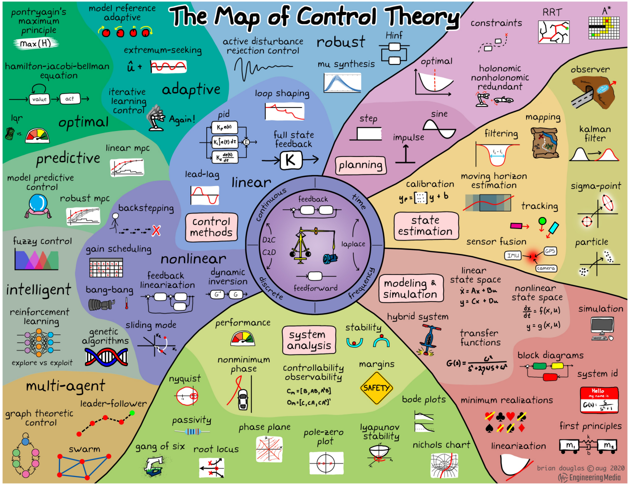


- Replace costly simulations by accurate simple model
- Preserve structure and properties (port-Hamiltonian)
- Allows for geometry optimization

M. Gouzien, C. P-V., G. Haine and D. Matignon, *"A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Forewords

The map of control theory (by Brian Douglas - `https://engineeringmedia.com/`)



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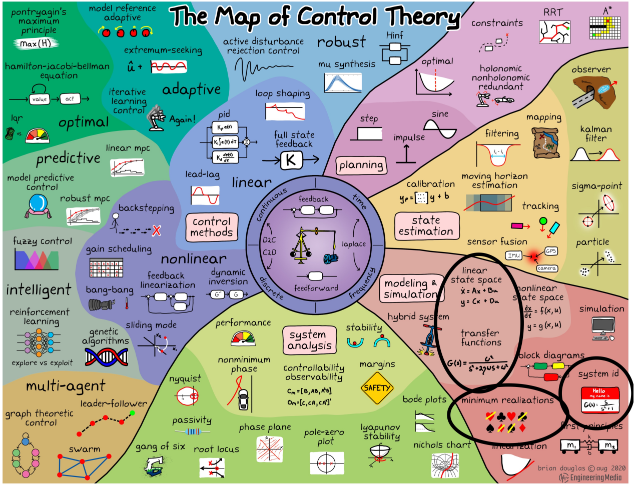

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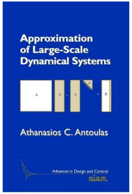

Antoulas/Beattie/Gugercin
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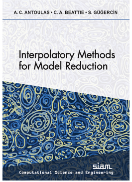

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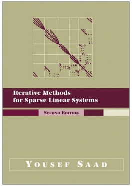

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- B-LTI, Q-LTI [Antoulas/Benner/Gosea/Karachalios/Pontes/Willcox/P-V./...]
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appli. Aircraft gust, vibration & flutter [Quero/Vuillemin/Reis/P-V.]
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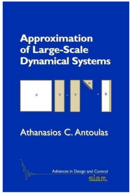

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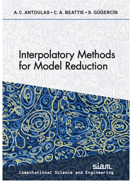

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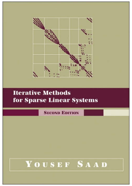

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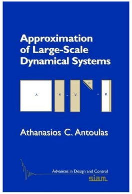

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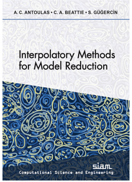

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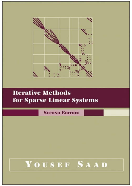

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Part 1 (reminder)

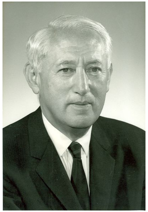
- Linear dynamical systems
- Realization and transfer functions

Part 2 (Loewner)

- Realization minimality
- Data-driven approximation
- Barycentric form

Part 3 (Loewner extended)

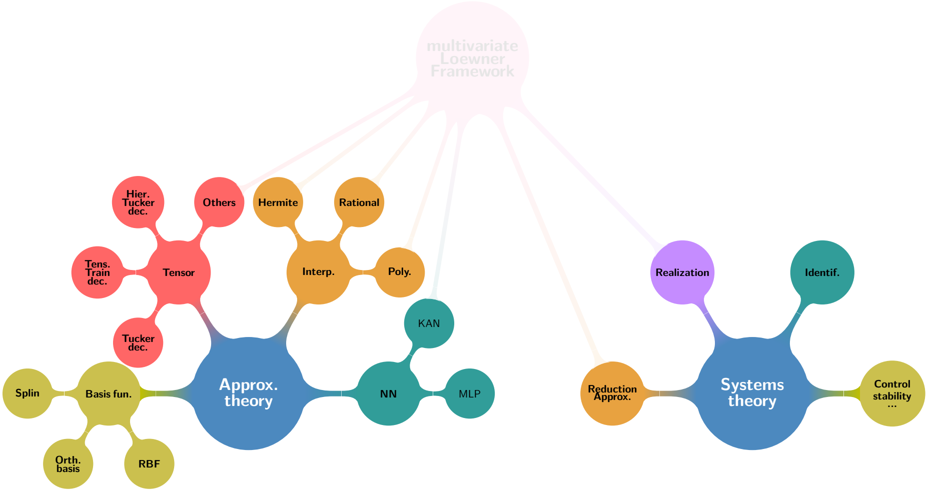
- Linear passive model (& pH)
- Linear parametric model



Karel Löwner (Czech)
1893 - 1968
Ph.D. advisor: G.A. Pick

Forewords

Approximation & systems theory... where we stand



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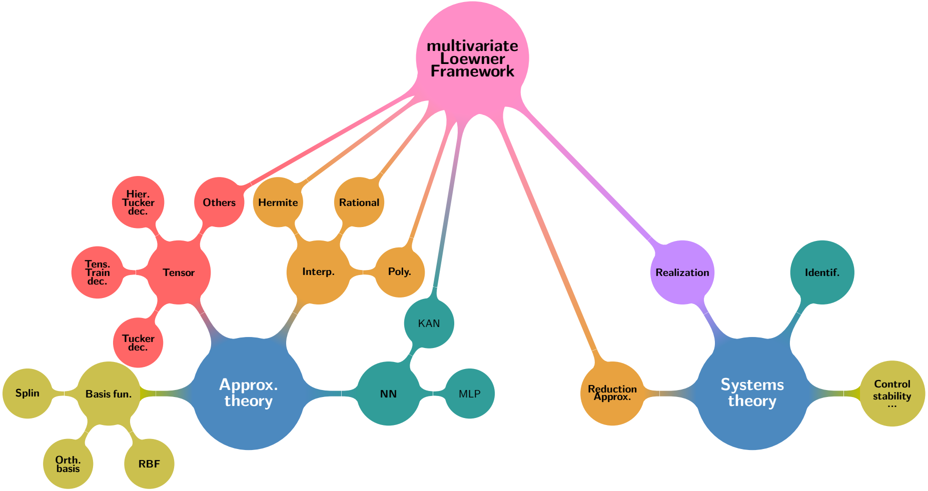
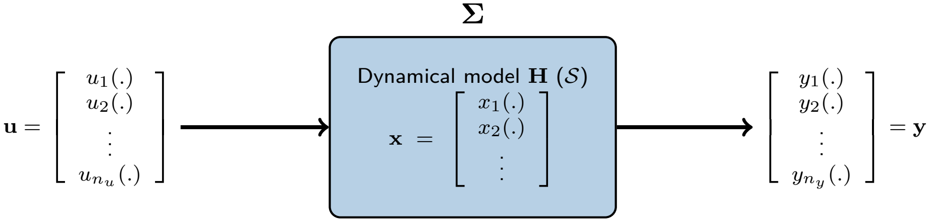


Content

Linear dynamical systems

Simplifications

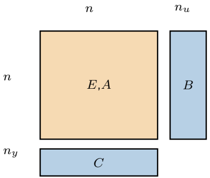
A dynamical model $\mathbb{H}$ (or $\mathcal{S}$) is a function mapping input $\mathbf{u}$ to output $\mathbf{y}$ signals of a system Σ



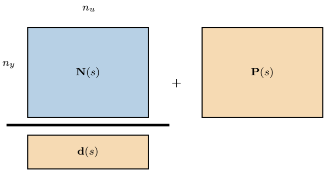
Let us stick (mainly) to linear systems only

Linear dynamical systems

Realization and transfer functions



$$\mathcal{S}:\left\{\begin{array}{rcl} E\dot{\mathbf{x}}(t) &=& A\mathbf{x}(t)+B\mathbf{u}(t)\\ \mathbf{y}(t) &=& C\mathbf{x}(t)\end{array}\right.$$



$$\begin{array}{rcl}\mathbf{H}(s) &=& C\left(sE-A\right)^{-1}B\\ &=& \mathbf{N}(s)/\mathbf{d}(s)+\mathbf{P}(s)\end{array}$$

Realizations

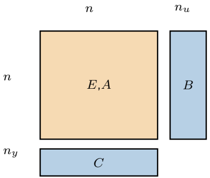
- E, A, B, C are (real) matrices
- Internal knowledge $\mathbf{u} \mapsto \mathbf{x} \mapsto \mathbf{y}$
- Realizations are infinite
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$
  $\mathbf{y}(t) \in \mathbb{R}^{n_y}$
  $\mathbf{x}(t) \in \mathbb{R}^{n}$

Transfer functions

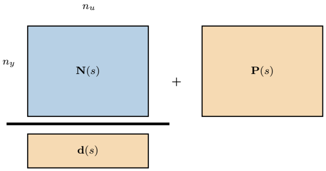
- $\mathbf{H}$ is a (complex) function
- External knowledge $\mathbf{u} \mapsto \mathbf{y}$
- Transfer functions are unique
- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$
  $\mathbf{y}(s) \in \mathbb{C}^{n_y}$

Linear dynamical systems

Realization and transfer functions



$$\mathcal{S} : \left\{ \begin{array}{rcl} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) \end{array} \right.$$



$$\begin{array}{rcl} \mathbf{H}(s) & = & C\Big(sE - A\Big)^{-1}B \\ & = & \mathbf{N}(s)/\mathbf{d}(s) + \mathbf{P}(s) \end{array}$$

Realizations

- E, A, B, C are (real) matrices
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- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$
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Transfer functions

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- $\mathbf{u}(s) \in \mathbb{C}^{n_u}$
  $\mathbf{y}(s) \in \mathbb{C}^{n_y}$

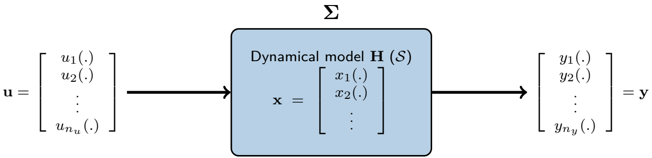
Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



Structures

- **L-ODE**
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1}$$

ODE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x \end{array}$$

Singularities λ and zeros z

$$\begin{array}{rcl} \lambda_{\mathcal{S}} & = & \texttt{eig(A,E)} \\ & = & \Lambda(-1,1) \\ & = & \{-1\} \\ z_{\mathcal{S}} & = & \texttt{eig([A B; C D],blkdiag(E,zeros(ny,nu))} \\ & = & \Lambda\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \\ & = & \{\infty, \infty\} \end{array}$$

Linear dynamical systems

Linear finite dimensional models

Structures

- L-ODE
- **L-ODE / DAE-1**
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x + 2u \\ y & = & x + 2u \end{array}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}_1} = \Lambda(-1, 1) = \{-1\}$$

Structures

- L-ODE
- **L-ODE / DAE-1**
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + 2u \\ 0 & = & -x_2 + 2u = x_2 - 2u \\ y & = & x_1 + x_2 \end{array}$$

Singularities of matrix pencil (A, E)[a]

$$\lambda_{\mathcal{S}_2} = \Lambda\left(\left[\begin{array}{c|c} -1 & \\ \hline & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]\right) = \{-1, \infty\}$$

[a] $B^\top = \begin{bmatrix} 2 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + 2 = \frac{2s+4}{s+1}$$

DAE index-1 realization $\mathcal{S}_2$ (canonical form)

$$\left(\left[\begin{array}{c|c} A_1 = -1 & \\ \hline & I_{n_2} = 1 \end{array} \right], \left[\begin{array}{c|c} I_{n_1} = 1 & \\ \hline & N = 0 \end{array} \right] \right)$$

Index is the k-nilpotent degree of N

- Finite dynamic modes
  $n_1 = 1$
- Infinite dynamic (impulsive) modes
  $\mathbf{rank}(E) - n_1 = \mathbf{rank}(N) = 1 - 1 = 0$
- Non dynamic modes
  $n - \mathbf{rank}(E) = 2 - 1 = 1$

Structures

- L-ODE
- L-ODE / DAE-1
- **L-DAE**
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}} = \{-1, \infty, \infty\}$$

- Finite dynamic modes $n_1 = 1$
- Impulsive modes $\mathbf{rank}(E) - n_1 = 2 - 1 = 1$
- Non dynamic modes $n - \mathbf{rank}(E) = 3 - 2 = 1$

Structures
- L-ODE
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- L-DDE
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Transfer function

$$\mathbf{H}(s) = \frac{2}{s+1} + s = \frac{s^2+s+2}{s+1}$$

DAE index-2 realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x}_2 & = & x_1 \\ \dot{x}_3 & = & x_2 \\ x_2 & = & -x_3 + u = x_3 - u \\ y & = & x_1 + x_2 + 2x_3 \end{array}$$

Singularities of matrix pencil (A, E)

$$\lambda_{\mathcal{S}} = \{-1, \infty, \infty\}$$

- Finite dynamic modes $n_1 = 1$
- Impulsive modes $\mathbf{rank}(E) - n_1 = 2 - 1 = 1$
- Non dynamic modes $n - \mathbf{rank}(E) = 3 - 2 = 1$

Linear dynamical systems

Linear finite dimensional models (about observability, controllability, minimality...)

Structures

- **L-ODE**
- **L-ODE / DAE-1**
- **L-DAE**
- L-DDE
- L-PDE

- Reachability matrix

$$\mathcal{R}_n(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$$

- Observability matrix

$$\mathcal{O}_n(A,C) = \begin{bmatrix} C^\top & A^\top C^\top & (A^\top)^2C^\top & \cdots & (A^\top)^nC^\top \end{bmatrix}^\top$$

- Minimality: both controllable and observable.
- Connection with Markov parameters

$$H_0 = D \ , \ H_k = CA^{k-1}B \ (k > 1)$$

$$\mathcal{O}_n\mathcal{R}_n = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & & \ddots & \vdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{bmatrix}$$

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- **L-DDE**
- L-PDE

Transfer function

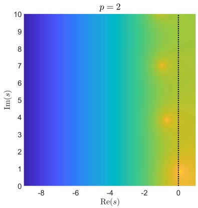
$$\mathbf{H}(s) = \frac{1}{s + e^{-ps}}$$

L-DDE realization $\mathcal{S}$

$$\begin{array}{rcl} \dot{x} & = & -x(t-p) + u \\ y & = & x \end{array}$$

Singularities (periodic)

$$\lambda_{\mathcal{S}} = \{\omega \text{ s.t. } s + \cos(s) - \imath \sin(s) = 0\}$$



Structures

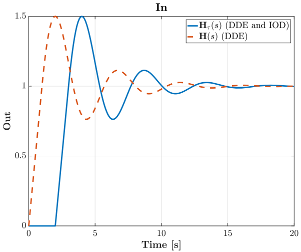
- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Transfer function

$$\mathbf{H}_\tau(s) = \frac{1}{s + e^{-ps}} e^{-2s}$$

L-DDE realization $\mathcal{S}$

$$\begin{aligned} \dot{x} &= -x(t-p) + u(t-2) \\ y &= x \end{aligned}$$

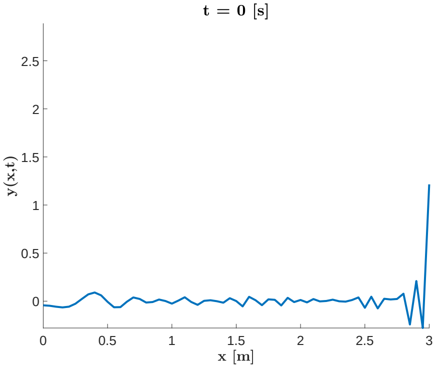


Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$



Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- **L-PDE**

Transfer function (boundary controlled transport)

$$\mathbf{H}(s) = \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s}$$

L-PDE "realization" $\mathcal{S}$

$$\begin{array}{rcl} \dfrac{\partial \tilde{y}(x,t)}{\partial x} + 2x \dfrac{\partial \tilde{y}(x,t)}{\partial t} & = & 0 \\ \tilde{y}(x,0) & = & 0 \\ \tilde{y}(0,t) & = & \dfrac{1}{\sqrt{t}} \star \tilde{u}_f(0,t) \\ \dfrac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0,s) & = & u_f(0,s) \end{array}$$

Singularities

$$\lambda_{\mathcal{S}} = \{0, \lambda_1, \overline{\lambda_1}\}$$

Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$

Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
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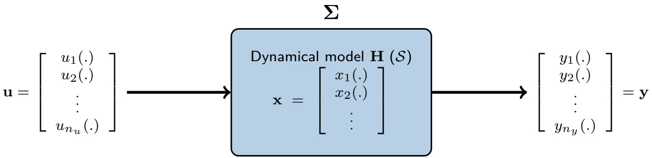
Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



Linear dynamical systems

Why all this? What is common?

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

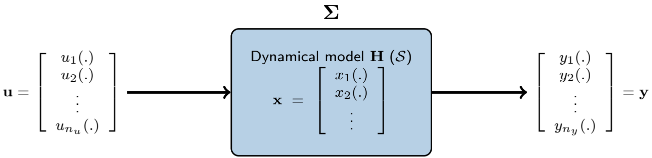
Data

- Time-domain
- **Frequency-domain**

Model

(Time-domain) $\mathcal{S} \sim \mathbf{u} \rightarrow \mathbf{x} \rightarrow \mathbf{y}$

(Frequency-domain) $\mathbf{H} \sim \mathbf{u} \rightarrow \mathbf{y}$



(Time-domain) $\{t_i, \mathbf{G}(t_i)\}_{i=1}^{N}$

(Frequency-domain) $\{z_i, \mathbf{G}(z_i)\}_{i=1}^{N}$

Data

Content

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \dots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \dots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \dots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

Loewner Matrix

Given a row / left array of pairs of complex numbers $\{\mu_i, \mathbf{v}_i\}$, $i = 1, \cdots, q$, and a column / right array of pairs of complex numbers $\{\lambda_j, \mathbf{w}_j\}$, $j = 1, \cdots, k$ with μ_i and λ_j distinct, the associated Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}.$$

If there is an underlying function $\mathbf{H}$ then

$$\mathbf{w}_j = \mathbf{H}(\lambda_j) \text{ and } \mathbf{v}_i = \mathbf{H}(\mu_i).$$

We will present the Loewner framework in connection with

(i) rational interpolation and consequently in connection with

(ii) reduced-order modeling of linear dynamical systems given frequency domain data.

The Loewner matrix rank encodes the minimal complexity of the solutions of the interpolation problem.

$$P := \{(x_i, y_i) \ : \ i = 1, \cdots, N, \ x_i \neq x_j, \ i \neq j\}$$

The idea behind the present approach to rational interpolation is to use a formula for rational interpolants which is similar to the one defining the Lagrange polynomial. First we partition the array P in two disjoint subarrays:

$$\begin{array}{lll} P_c & := & \{(\lambda_j, \mathbf{w}_j) \ j = 1, \ldots, k\} \\ P_r & := & \{(\mu_i, \mathbf{w}_i) \ i = 1, \ldots, q\} \end{array}$$

e.g. $N = k + q$

$$\begin{array}{lll} \lambda_j = x_j & \mathbf{w}_j = y_j & j = 1, \ldots, k \\ \mu_i = x_{k+i} & \mathbf{v}_i = y_{k+i} & i = 1, \ldots, q \end{array}$$

Using array P_c , for constants α_j and $\mathbf{w}_j$, consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^{k} \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0 \ , \ \alpha_j \neq 0$$

For constants α_j and $\mathbf{w}_j$, consider $\mathbf{H}(s)$ given by

$$\sum_{j=1}^{k} \alpha_j \frac{\mathbf{H}(s) - \mathbf{w}_j}{s - \lambda_j} = 0 \ , \ \alpha_j \neq 0$$

solving for $\mathbf{H}$, we get

$$\mathbf{H}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s - \lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s - \lambda_j}}$$

It follows that

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j$$

ensures P_c constraints by construction. This is the barycentric (rational) Lagrange interpolation formula.

The free parameters α_j can be determined so that the additional constraints contained in array P_r are satisfied:

$$\mathbf{H}(\mu_i) = \mathbf{v}_i$$

Then it follows $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^\top - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^\top - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k.$$

Lemma

Given the rational function $\mathbf{H}$ and an array of points P, where $y_j = \mathbf{H}(x_j)$ and x_j is not a pole of $\mathbf{H}$, let $\mathbb{L}$ be a $q \times k$ Loewner matrix, for some partitioning P_c, P_r of P. Then

$$q, k \geq \mathbf{deg}(\mathbf{H}) \Rightarrow \mathbf{rank}\, \mathbb{L} = \mathbf{deg}(\mathbf{H})$$

Loewner

Scalar rational interpolation and the Loewner matrix (a rational Lagrange-type formula)

$$\begin{cases} P_c & := & \{(\lambda_j; \mathbf{w}_j)\,,\ j=1,\dots,k\} \\ P_r & := & \{(\mu_i; \mathbf{v}_i)\,,\ i=1,\dots,q\} \end{cases}$$

Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q\times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

$$M\mathbb{L} - \mathbb{L}\Lambda = V - W$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\sum_{j=1}^{k} \frac{\alpha_j \mathbf{w}_j}{s-\lambda_j}}{\sum_{j=1}^{k} \frac{\alpha_j}{s-\lambda_j}}$$

Null space

$$\mathbf{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{C}^k$$

Data generated from $\mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2), with $k=3$, $q=4$

$$\left.\begin{array}{lcl} \lambda_j & = & [1,3,5] \\ \mu_i & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{lcl} \mathbf{w}_j & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_i & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix} \in \mathbb{C}^{4\times 3}$$

Null space

$$\textbf{span}(\mathbf{c}) = \mathcal{N}(\mathbb{L})$$

$$\mathbf{c} = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Consider system $\mathbf{H}$ in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{, where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \dots, \lambda_k\}, \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ and solving

$$\mathbb{L}\mathbf{c} = 0$$

leads to $\mathbf{H}$ in Lagrangian basis

$$\mathbf{H}(s) = \underbrace{\mathbf{c}^\top \mathbf{w}}_{C} \underbrace{\begin{bmatrix} \mathbf{L}_{s,\lambda,k} \\ \mathbf{c}^\top \end{bmatrix}^{-1}}_{\Phi(s)^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s-\lambda_1 & \lambda_2 - s & \\ s-\lambda_1 & & \lambda_3 - s \\ \vdots & & & \ddots \end{bmatrix} \in \mathbb{R}^{n \times (n+1)} \text{ and } \Phi(s) \in \mathbb{C}^{(n+1)\times(n+1)}$$

Consider system $\mathbf{H}$ in barycentric form

$$\mathbf{H}(s) = \frac{\sum_i^n \beta_i \mathbf{q}_i(s)}{\sum_i^n \alpha_i \mathbf{q}_i(s)} \text{ , where } \mathbf{q}_i(s) = \prod_{i'=1, i' \neq i}^{n} (s - \lambda_{i'})$$

Constructing the Loewner matrix with $\{\lambda_1, \ldots, \lambda_k\}, \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and solving

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leads to $\mathbf{H}$ in Lagrangian basis

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$$\mathbf{L}_{s,\lambda,n} = \begin{bmatrix} s - \lambda_1 & \lambda_2 - s & \\ s - \lambda_1 & & \lambda_3 - s \\ \vdots & & \ddots \end{bmatrix} \in \mathbb{R}^{n \times (n+1)} \text{ and } \Phi(s) \in \mathbb{C}^{(n+1)\times(n+1)}$$

Loewner

Loewner barycentric examples (simple case, `CAS=1`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\mathbb{L} = \begin{bmatrix} \frac{\frac{2}{3}-1}{2-1} & \frac{\frac{2}{3}-\frac{1}{2}}{2-3} \\ \frac{\frac{2}{5}-1}{4-1} & \frac{\frac{2}{5}-\frac{1}{2}}{4-3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

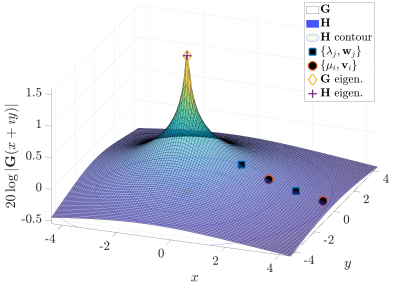
Loewner

Loewner barycentric examples (simple case, `CAS=1`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 2$, $\mathbf{H}(s) = C\mathbf{\Phi}(s)^{-1}B = \mathbf{G}(s)$



$$\mathbf{G}(s) = \frac{2}{s+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 3 \text{ and } \mu_1 = 2, \mu_2 = 4$$

Leads to

$$\mathbf{w}_1 = 1, \mathbf{w}_2 = \frac{1}{2} \text{ and } \mathbf{v}_1 = \frac{2}{3}, \mathbf{v}_2 = \frac{2}{5}$$

$$\ker(\mathbb{L}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{c}$$

$$C = \mathbf{c}^\top \mathbf{w} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \mathbf{L}_{s,\lambda,1} \\ \mathbf{c}^\top \end{bmatrix} = \begin{bmatrix} s-1 & 3-s \\ -1 & 2 \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\mathbb{L} = \begin{bmatrix} -\frac{1}{5} + \frac{3}{5}\imath & -\frac{1}{34} + \frac{13}{34}\imath & \frac{5}{17} + \frac{3}{17}\imath \\ \frac{17}{290} + \frac{59}{290}\imath & \frac{57}{986} + \frac{75}{986}\imath & \frac{31}{986} - \frac{63}{986}\imath \\ -\frac{46}{265} + \frac{108}{265}\imath & -\frac{37}{901} + \frac{209}{901}\imath & \frac{118}{901} + \frac{64}{901}\imath \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization $n = 3$

$$C = \begin{bmatrix} c_1\mathbf{w}_1 & c_2\mathbf{w}_2 & c_3\mathbf{w}_3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} s - \imath & 2\imath - s & \\ s - \imath & & -2 + \imath - s \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Resulting in

$$\frac{-\imath}{-s^2\imath + s(1-\imath) + (1-\imath)} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2 + \imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

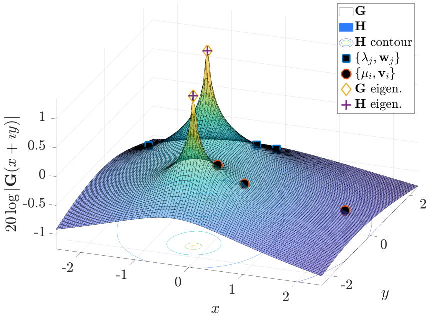
$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

$$\ker(\mathbb{L}) = \begin{bmatrix} 13/17 - 16\imath/17 \\ -2 + 2\imath \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Loewner barycentric examples (complex dynamical tippe top case, `CAS=3`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
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Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

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Code break (https://github.com/cpoussot/lf)

```
% Model (CAS=1)
S       = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses
la  = [1 3]; k  = length(la);
mu  = [2 4]; q  = length(mu);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
LL  = lf.loewnerMatrix(la,mu,W,V);
c   = null(sym(LL));
% Realization
C   = c.'.*W(:).';
B   = zeros(k,1); B(end) = 1;
PHI = [s-la(1)*ones(k-1,1) diag(la(2:end))-s*eye(k-1); c.'];
Hr  = simplify(C*(PHI\B));
```

Loewner

Code break (https://github.com/cpoussot/lf)

```
% Model (CAS=1)
S     = ss(tf(2,[1  1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses
la    = [1  3]; k     = length(la);
mu    = [2  4]; q     = length(mu);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
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```

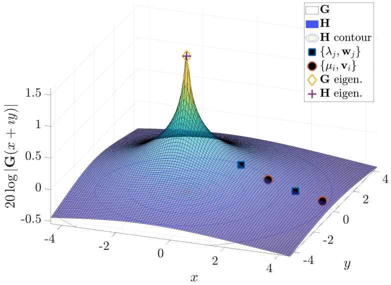

SISO interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{w}_j\} & j = 1, \dots, k \\ \{\mu_i, \mathbf{v}_i^\top\} & i = 1, \dots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j) & = & \mathbf{w}_j & j = 1, \dots, k \\ \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \dots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

MIMO tangential interpolation problem

Given the right and left data (λ_j and μ_i are distinct):

$$\begin{array}{ll} \{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\} & j = 1, \ldots, k \\ \{\mu_i, \mathbf{l}_i^\top, \mathbf{v}_i^\top\} & i = 1, \ldots, q \end{array}$$

we seek $\mathcal{S} : (E, A, B, C)$, whose transfer function is $\mathbf{H}(s) = C(sE - A)^{-1}B$ s.t.

$$\begin{array}{rcll} \mathbf{H}(\lambda_j)\mathbf{r}_j & = & \mathbf{w}_j & j = 1, \ldots, k \\ \mathbf{l}_i^\top \mathbf{H}(\mu_i) & = & \mathbf{v}_i^\top & i = 1, \ldots, q \end{array}$$

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

I.V. Gosea, C. P-V. and A.C. Antoulas, *"Data-driven modeling and control of large-scale dynamical systems in the Loewner framework"*, Handbook in Numerical Analysis, vol. 23, January 2022.

The right data can be expressed as:

$$\begin{array}{rcl} \Lambda & = & \textbf{diag}\,[\lambda_1,\ldots,\lambda_k] \in \mathbb{C}^{k\times k}, \\ \mathbf{R} & = & \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \ldots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{n_u\times k} \\ \mathbf{W} & = & \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y\times k} \end{array}$$

and the left data can be expressed as:

$$\begin{array}{rcl} \mathbf{M} & = & \textbf{diag}\,[\mu_1,\ldots,\mu_q] \in \mathbb{C}^{q\times q} \\ \mathbf{L}^\top & = & \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{n_y\times q} \\ \mathbf{V}^\top & = & \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{n_u\times q} \end{array}$$

The Loewner matrix in this case is

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1-\lambda_1} & \ldots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q-\lambda_1} & \ldots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

With this notation $\mathbb{L}$ satisfy the Sylvester equation : $\mathbf{M}\mathbb{L} - \mathbb{L}\Lambda = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$.

The right data can be expressed as:

$$
\begin{aligned}
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\mathbf{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{n_y\times k}
\end{aligned}
$$

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$$
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\mathbf{M} &= \mathbf{diag}\,[\mu_1, \ldots, \mu_q] \in \mathbb{C}^{q\times q}\\
\mathbf{L}^\top &= \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \ldots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{n_y\times q}\\
\mathbf{V}^\top &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{n_u\times q}
\end{aligned}
$$

The Loewner matrix in this case is

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\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1}{\mu_1 - \lambda_1} & \ldots & \frac{\mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1}{\mu_q - \lambda_1} & \ldots & \frac{\mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}
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The shifted Loewner matrix is:

$$\mathbb{M} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_1 - \mathbf{l}_1^\top \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{v}_1^\top \mathbf{r}_k - \mathbf{l}_1^\top \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_1 - \mathbf{l}_q^\top \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \dots & \frac{\mu_q \mathbf{v}_q^\top \mathbf{r}_k - \mathbf{l}_q^\top \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

$$\mathbf{M}\mathbb{M} - \mathbb{M}\mathbf{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\mathbf{\Lambda}$$

Loewner

Loewner pencil

If data are sampled from $\mathbf{G}(s) = C(sE-A)^{-1}B$, let us define :

$$\mathcal{O}_q = \begin{bmatrix} \mathbf{l}_1^\top C(\mu_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_q^\top C(\mu_q E - A)^{-1} \end{bmatrix} \in \mathbb{C}^{q\times n}, \quad \mathcal{R}_k = \left[(\lambda_1 E - A)^{-1}B\mathbf{r}_1, \dots, (\lambda_k E - A)^{-1}B\mathbf{r}_k,\right] \in \mathbb{C}^{n\times k}$$

be the generalized tangential observability and controllability matrices. Then,

$$\begin{aligned} [\mathbb{L}]_{ij} &= \frac{\mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} E (\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i E [\mathcal{R}_k]_j \end{aligned}$$

$$\begin{aligned} [\mathbb{M}]_{ij} &= \frac{\mu_i \mathbf{v}_i^\top \mathbf{r}_j - \mathbf{l}_i^\top \mathbf{w}_j \lambda_j}{\mu_i - \lambda_j} \\ &= -\mathbf{l}_i^\top C(\mu_j E - A)^{-1} A (\lambda_i E - A)^{-1} B\mathbf{r}_j \\ &= -[\mathcal{O}_q]_i A [\mathcal{R}_k]_j \end{aligned}$$

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Assume that $k = q$, then $\mathbf{H}(s) = C(sE - A)^{-1}B$ with

$$E = -\mathbb{L}, \quad A = -\mathbb{M}, \quad , B = \mathbf{V}, \quad C = \mathbf{W},$$

is **a descriptor realization** interpolating the data.

Suppose that we have more data than necessary. The problem has a solution if

$$\mathbf{rank}[\xi\mathbb{L} - \mathbb{M}] = \mathbf{rank}[\mathbb{L},\ \mathbb{M}] = \mathbf{rank}\begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = r, \quad \xi \in \{\lambda_i\} \cup \{\mu_j\}$$

$$[\mathbb{L},\ \mathbb{M}] = Y\Sigma_l\tilde{X}^\top, \quad \begin{bmatrix}\mathbb{L}\\ \mathbb{M}\end{bmatrix} = \tilde{Y}\Sigma_r X^H, \ Y, X \in \mathbb{C}^{N\times n}.$$

A realization (E, A, B, C) of an (approximate) interpolant is given by:

$$E = -Y^H\mathbb{L}X, \quad A = -Y^H\mathbb{M}X, B = -Y^H\mathbf{V}, C = \mathbf{W}X$$

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Being known the full state realization $\mathcal{S}$

$$\begin{array}{rcl} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) + D\mathbf{u}(t) \end{array}$$

The Petrov-Galerkin reduced order model $\hat{\mathcal{S}}$ is obtained with projectors $W, V \in \mathbb{C}^{n \times r}$ as

$$\begin{array}{rcl} W^\top EV\dot{\hat{\mathbf{x}}}(t) & = & W^\top AV\hat{\mathbf{x}}(t) + W^\top B\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) & = & CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{array}$$

The above equations assume $W = \mathbf{span}\,(\mathcal{W}), V = \mathbf{span}\,(\mathcal{V})$ selected such that they meet the approximation objective:

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t), \quad , \hat{\mathbf{x}}_0 = W^\top \mathbf{x}_0$$

Choosing two different bases V' and W' that respectively span the same subspaces $\mathcal{V}$ and $\mathcal{W}$ result in the same reconstructed solution $\mathbf{x}(t)$. Thus, subspaces are relevant, not basis.

A.J. Mayo and A.C. Antoulas, *"A framework for the solution of the generalized realization problem"*, Linear Algebra and its Applications, vol. 425(2-3), 2007.

Now recall,

$$\mathcal{O}E\mathcal{R} = \mathbb{L} \quad \text{and} \quad \mathcal{O}A\mathcal{R} = \mathbb{M},$$

where the generalized reachability and observability matrices are (complex conjugation trick for realness)

$$\mathcal{R} = \begin{bmatrix} \mathbf{r}_1^\top B^\top (\lambda_1 E - A)^{-\top} \\ \overline{\mathbf{r}}_1^T B^\top (\overline{\lambda}_1 E - A)^{-\top} \\ \vdots \\ \mathbf{r}_{k/2}^\top B^\top (\lambda_{k/2} E - A)^{-\top} \\ \overline{\mathbf{r}}_{k/2} B^\top (\overline{\lambda}_{k/2} E - A)^{-\top} \end{bmatrix}^\top \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{l}_1 C(\mu_1 E - A)^{-1} \\ \overline{\mathbf{l}}_1 C(\overline{\mu}_1 E - A)^{-1} \\ \vdots \\ \mathbf{l}_{q/2} C(\mu_{q/2} E - A)^{-1} \\ \overline{\mathbf{l}}_{q/2} C(\overline{\mu}_{q/2} E - A)^{-1} \end{bmatrix}.$$

$$(Y^\top J^H \mathcal{O})^\top E(\mathcal{R}JX) = \mathbb{L}, \quad \text{and} \quad (Y^\top J^H \mathcal{O})^\top A(\mathcal{R}JX) = \mathbb{M},$$

$$J_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ 1 & -\imath \end{bmatrix} \quad \text{and} \quad J = I_{k/2} \otimes (J_0 \otimes I_m),$$

M. Gouzien, C. P-V., G. Haine and D. Matignon, *"A Port-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Therefore,

$$\underbrace{(Y^\top J^H \mathcal{O})^\top}_{W^\top} E \underbrace{(\mathcal{R} J X)}_{V} = \mathbb{L}, \quad \text{and} \quad \underbrace{((Y^\top J^H \mathcal{O})^\top}_{W^\top} A \underbrace{(\mathcal{R} J X)}_{V} = \mathbb{M},$$

Therefore,

$$W = Y^\top J^H \mathcal{O} \quad \text{and} \quad V = \mathcal{R} J X.$$

and

$$\mathbf{x}(t) \approx V \hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R} J X.$$

can be recovered (approximated) solely from ROM simulation.

M. Gouzien, C. P-V., G. Haine and D. Matignon, *"APort-Hamiltonian reduced order modelling of the 2D Maxwell equations"*, journal for Computation and Mathematics in Electrical and Electronic Engineering, 2025.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Given $\{\lambda_j, \mathbf{r}_j, \mathbf{w}_j\}$ and $\{\mu_i, \mathbf{l}_i, \mathbf{v}_i\}$, seek $\mathbf{H}$ s.t.

$$\mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \text{ and } \mathbf{l}_i\mathbf{H}(\mu_i) = \mathbf{v}_i$$

$j = 1, \ldots, k$; $i = 1, \ldots, q$.

Rational interpolation
$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$

- underlying rational (r) order

$$\begin{array}{rcl} r & = & \mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) \\ & = & \mathbf{rank}([\mathbb{L}, \mathbb{M}]) \\ & = & \mathbf{rank}([\mathbb{L}^H, \mathbb{M}^H]^H) \end{array}$$

- and McMillan (ν) order

$$\nu \quad = \quad \mathbf{rank}(\mathbb{L})$$

- $\mathbb{L}$ and $\mathbb{M}$ are input-output independents.
- Minimal realization
- If $\mathcal{S}$ is known, then internal state $\mathbf{x} \in \mathbb{R}^n$ is recovered by

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}_r(t) \quad \text{where} \quad V = \mathcal{R}JX.$$

A.C. Antoulas, S. Lefteriu and A.C. Ionita, *"Chapter 8: A Tutorial Introduction to the Loewner Framework for Model Reduction"*, Model Reduction and Approximation: Theory and Algorithms, 2016.

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

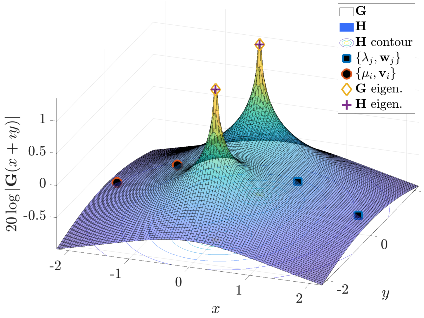
$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
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Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$, $(\mathbb{M}, \mathbb{L})$ pencil regular

$$\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V} = \mathbf{G}(s)$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

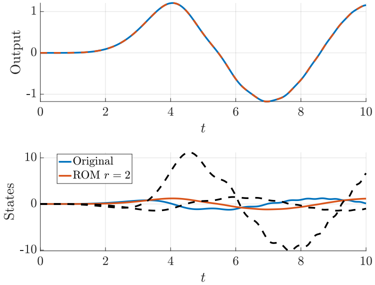
$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal variables



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

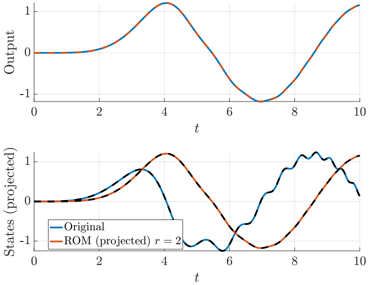
$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Output and internal **projected** variables



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{w}_1 = \frac{1}{2}, \mathbf{w}_2 = \frac{1}{5}$$
$$\mathbf{v}_1 = \frac{1}{2}, \mathbf{v}_2 = \frac{1}{5}.$$

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

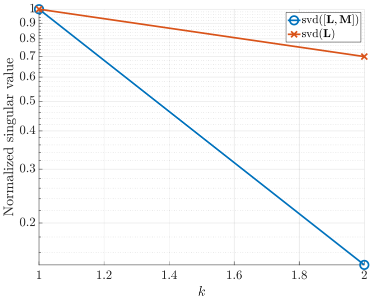
Loewner

Loewner examples (rectangular case, `CAS=5`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Realization rect.: $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{\dagger}\mathbf{V}$



$$\mathbf{G}(s) = \frac{1}{s^2+1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \end{bmatrix}$$

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$r = 2$ and $\nu = 2$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{\dagger}\mathbf{V} \\ &= \frac{1}{s^2 - 4.650e - 16s + 1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + 1}$$

Evaluated at

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_2 = 3$$
$$\mu_1 = -1, \mu_2 = -2$$

Leads to

$$\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} 0 & -\frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & 0 & -\frac{1}{50} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \end{bmatrix}$$

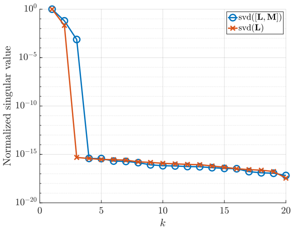
Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

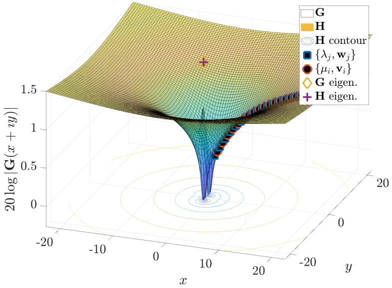
$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

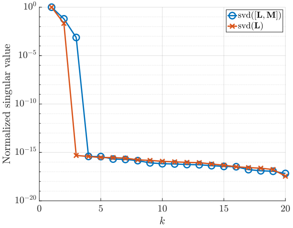
Realization $n = 20$: $\mathbf{H}(s) = \mathbf{W}(-s\mathbb{L} + \mathbb{M})^{-1}\mathbf{V}$



$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\dots 20} = [1, 2, \dots, 20]$$
$$\mu_{1\dots 20} = [1.5, 2.5, \dots, 20.5]$$



Loewner

Loewner examples (lot of data case, `CAS=6`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

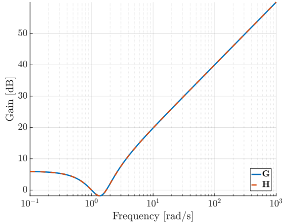
$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r = 3$$

$$\mathbf{rank}(\mathbb{L}) = \nu = 2$$

$$\mathbf{G}(s) = \frac{s^2 + s + 2}{s + 1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Loewner

Loewner examples (lot of data case, `CAS=6`)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r = 3$$

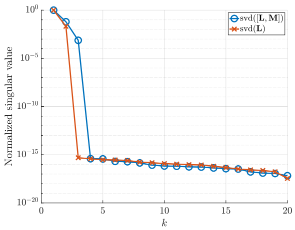
$$\mathbf{rank}(\mathbb{L}) = \nu = 2$$

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{W}X(-sY^\top\mathbb{L}X + Y^\top\mathbb{M}X)^{-1}Y^\top\mathbf{V} \\ &= \frac{s^2+s+2}{s+1}\end{aligned}$$

$$\mathbf{G}(s) = \frac{s^2+s+2}{s+1}$$

Evaluated at

$$\lambda_{1\ldots20} = [1, 2, \ldots, 20]$$
$$\mu_{1\ldots20} = [1.5, 2.5, \ldots, 20.5]$$



Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5-\imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

Loewner examples (complex dynamical tippe top case, CAS=3)

Rational function satisfies

$$\mathbf{H}(\lambda_j) = \mathbf{w}_j \text{ and } \mathbf{H}(\mu_i) = \mathbf{v}_i$$

Rank reveals the underlying rational (r) and McMillan (ν) orders

$$\mathbf{rank}(\xi\mathbb{L} - \mathbb{M}) = r$$

$$\mathbf{rank}(\mathbb{L}) = \nu$$

$$\mathbf{H}(s) = \frac{(1 + 2.22e - 16\imath)}{s^2 + (1+\imath)s + (1+\imath)}$$

$$\mathbf{G}(s) = \frac{1}{s^2 + (1+\imath)s + (1+\imath)}$$

Evaluated at

$$\lambda_1 = \imath, \lambda_2 = 2\imath, \lambda_3 = -2+\imath$$
$$\mu_1 = -\imath, \mu_2 = 2, \mu_3 = 0.5 - \imath$$

Leads to

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$$

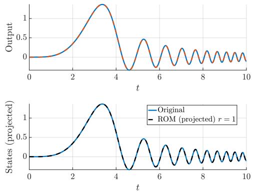
$$\hat{\mathbb{L}} = \begin{bmatrix} -0.207 + 0.9568\imath & -0.1276 - 0.0294\imath \\ -0.0438 - 0.0818\imath & 0.039 - 0.1089\imath \end{bmatrix}$$

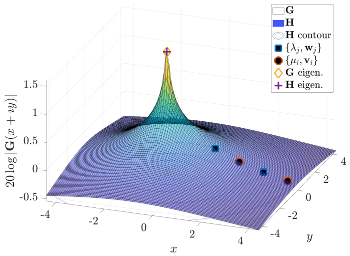
$$\hat{\mathbb{M}} = \begin{bmatrix} 0.4738 - 0.0560\imath & 0.0089 - 0.3392\imath \\ 0.2253 - 0.1607\imath & -0.0342 + 0.1308\imath \end{bmatrix}$$

```
% Model (CAS = 1)
S     = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses, tangent
la    = [1 3];
mu    = [2 4];
k     = length(la);
q     = length(mu);
R     = ones(1,k);
L     = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```

Code break (https://github.com/cpoussot/lf)

```
% Model (CAS = 1)
S     = ss(tf(2,[1 1])); S.E = 1;
[A,B,C,D,E] = dssdata(S);
G = @(s) C*((s*E-A)\B)+D;
% IP, responses, tangent
la    = [1 3];
mu    = [2 4];
k     = length(la);
q     = length(mu);
R     = ones(1,k);
L     = ones(q,1);
for ii = 1:k; W(1,1,ii) = G(la(ii)); end
for ii = 1:q; V(1,1,ii) = G(mu(ii)); end

% Loewner
[hr,info] = lf.loewner_tng(la,mu,W,V,R,L);
```





Content

Loewner extensions (passive & pH)

More structures and properties

Structures

L-ODE

L-ODE / DAE-1

L-DAE

L-DDE

L-PDE

L-pH

pL-DAE

B-DAE

Q-DAE

Model

- (A, B, C) and $\mathbf{H}(s)$
- (A, B, C, D) and $\mathbf{H}(s)$
- (E, A, B, C) and $\mathbf{H}(s)$
- $(A_i \ldots, B, C, \tau_i)$ and $\mathbf{H}(s)$
- $\mathbf{H}(s)$
- (Q, J, R, G, P, N, S) and $\mathbf{H}(s)$
- (E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
- (A, B, C, N) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$
- (A, B, C, Q) and $\mathbf{H}(s_1, s_2, \ldots, s_k)$

Loewner extensions (passive & pH)

More structures and properties

Structures	Model
L-ODE	(A, B, C) and $\mathbf{H}(s)$
L-ODE / DAE-1	(A, B, C, D) and $\mathbf{H}(s)$
L-DAE	(E, A, B, C) and $\mathbf{H}(s)$
L-DDE	$(A_i \dots, B, C, \tau_i)$ and $\mathbf{H}(s)$
L-PDE	$\mathbf{H}(s)$
L-pH	(Q, J, R, G, P, N, S) and $\mathbf{H}(s)$
pL-DAE	(E_j, A_j, B_j, C_j) and $\mathbf{H}(s, p_j)$
B-DAE	(A, B, C, N) and $\mathbf{H}(s_1, s_2, \dots, s_k)$
Q-DAE	(A, B, C, Q) and $\mathbf{H}(s_1, s_2, \dots, s_k)$

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Passivity

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u}\end{array}$$

$$\mathbf{\Phi_H}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$$

satisfy $\mathbf{\Phi_H}(\imath\omega) > 0$, $\texttt{Re}(\lambda_S) < 0$ and $D \succ 0$.

pH

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{array}$$

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Passivity

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} & = & C\mathbf{x} + D\mathbf{u}\end{array}$$

$$\mathbf{\Phi_H}(s) = \mathbf{H}(s) + \mathbf{H}^\top(-s)$$

satisfy $\mathbf{\Phi_H}(\imath\omega) > 0$, $\text{Re}(\lambda_S) < 0$ and $D \succ 0$.

pH

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{array}$$

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE
- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x+2u \\ y & = & x+2u \end{array}$$

Passivity

- $\mathtt{Re}(\Lambda(A)) = -1 < 0$
- $D = 2 \succ 0$
- $\mathbf{\Phi}_{\mathbf{H}}(\imath\omega) = \mathbf{H}(\imath\omega) + \mathbf{H}(-\imath\omega)^\top > 0$

$$\begin{array}{rcl} \mathbf{\Phi}_{\mathbf{H}}(s) & = & \frac{4\left(s^2-2\right)}{s^2-1} \\ \mathbf{\Phi}_{\mathbf{H}}(\imath\omega) & = & \frac{4\left(\omega^2+2\right)}{\omega^2+1} \end{array}$$

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Structures

- L-ODE
- L-ODE / DAE-1
- L-DAE
- L-DDE
- L-PDE

- **L-pH**
- pL-DAE
- B-DAE
- Q-DAE

Transfer function

$$\mathbf{H}(s) = \frac{2s+4}{s+1}$$

ODE realization $\mathcal{S}_1$

$$\begin{array}{rcl} \dot{x} & = & -x+2u \\ y & = & x+2u \end{array}$$

L-pH realization $\mathcal{S}_2$

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & (J-R)Q\mathbf{x}+(G-P)\mathbf{u} = (0-1)1\mathbf{x}+(-2-\sqrt{2})\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x}+(N+S)\mathbf{u} = (-2+\sqrt{2})\mathbf{x}+(0+2)\mathbf{u} \end{array}$$

where $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

The transfer function $\mathbf{H}(s)$ is

- **strictly passive** if it is **strictly positive-real** and **asymptotically** stable.
- **(non strictly) passive** if it is **positive-real** and **stable**.

The rational transfer function $\mathbf{H}(s)$ is **strictly positive-real** if $\mathbf{\Phi_H}(\imath\omega) > 0$ and **positive-real** if $\mathbf{\Phi_H}(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$, $\mathbf{\Phi_H}(\imath\omega) = \mathbf{H}(\imath\omega)^H + \mathbf{H}(\imath\omega)$

Real $X \succ 0$ is a (strict) passivity certificate for realization $\mathcal{S} : (A, B, C, D)$ iff.

$$W(X, \mathcal{S}) = \begin{bmatrix} -A^\top X - XA & C^\top - XB \\ (\star)^\top & D + D^\top \end{bmatrix} \succeq (\succ) 0$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

LTI pH model

LTI pH system model of a proper transfer function $\mathbf{H}$, has the state-space form

$$\begin{array}{rcl} \dot{\mathbf{x}} & = & (J-R)Q\mathbf{x} + (G-P)\mathbf{u} \\ \mathbf{y} & = & (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u} \end{array}$$

where

$$\mathcal{V} = \begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} \text{ and } \mathcal{W} = \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix}$$

satisfy $\mathcal{V} = -\mathcal{V}^\top$, $\mathcal{W} = \mathcal{W}^\top \succeq 0$ and $Q = Q^\top \succeq 0$

LTI strictly passive & LTI pH equivalence

If Q, $\mathcal{W}$ are invertible, $X = Q$ is a strict passivity certificate (*i.e.* $W(X, \mathcal{S}) \succ 0$) of the normalized pH system, $\mathcal{S} : (A, B, C, D)$, it follows that $\mathcal{S}$, can always be transformed in the pH realization.

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The right data can be expressed as:

$$\begin{aligned}
\Lambda &= \mathbf{diag}\left[\lambda_1, \dots, \lambda_k\right] \in \mathbb{C}^{k\times k}, \\
\mathbf{R} &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_k \end{bmatrix} \in \mathbb{C}^{m\times k} \\
\mathbf{W} &= \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_k \end{bmatrix} \in \mathbb{C}^{m\times k}
\end{aligned}$$

and the left data can be expressed as:

$$\begin{aligned}
\mathbf{M} &= -\Lambda^H &&= \mathbf{diag}\left[\mu_1, \dots, \mu_q\right] \in \mathbb{C}^{q\times q} \\
\mathbf{L}^\top &= \mathbf{R}^H &&= \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \dots & \mathbf{l}_q \end{bmatrix} \in \mathbb{C}^{m\times q} \\
\mathbf{V}^\top &= -\mathbf{W}^H &&= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{m\times q}
\end{aligned}$$

Spectral zeros are $(\lambda_j, \mathbf{r}_j)$ from standard Loewner (n zeros in the open right half-plane)

$$\begin{bmatrix} 0 & A & B \\ A^\top & 0 & C^\top \\ B^\top & C & D + D^\top \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix} = \lambda_j \begin{bmatrix} 0 & E & 0 \\ E^\top & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_j \\ q_j \\ \mathbf{r}_j \end{bmatrix}$$

A.C. Antoulas, *"A new result on passivity preserving model reduction"*, Systems & Control Letters, vol. 54, 2005.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

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Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Then $\mathbf{H}(s) \in \mathbb{C}^{m\times m}$ of McMillan degree n and $\mathcal{S}$ is a normalized pH form and satisfies

$$\mathbf{H}(\infty) = D \ , \ \mathbf{H}(\lambda_j)\mathbf{r}_j = \mathbf{w}_j \ , \ \mathbf{r}_j^H\mathbf{H}(-\overline{\lambda_j}) = -\mathbf{w}_j^H \ , \ D + D^\top \succ 0 \text{ and } \mathbb{L} \succ 0$$

By construction, one obtains an Hermitian $\mathbb{L} \in \mathbb{C}^{r\times r}$ and a skew symmetric $\mathbb{M} \in \mathbb{C}^{r\times r}$ matrix. By setting, $\mathbf{H}(\infty) = D$, one recovers an $m \times m$ real transfer function $\hat{\mathbf{H}}$. As $\mathbb{L} \succ 0$, one may apply the Cholesky decomposition $\mathbb{L} = T^\top T$. Then the *normalized pH model* is obtained as $\boldsymbol{\Sigma}_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, D)$,

$$\mathbf{S} := \begin{bmatrix} -T\hat{A}T^{-1} & -T\hat{B} \\ \hat{C}T^{-1} & D \end{bmatrix},$$

one obtains the equivalent pH-form by solving

$$\begin{bmatrix} -J & -G \\ G^\top & N \end{bmatrix} := \frac{\mathbf{S} - \mathbf{S}^\top}{2} \text{ and } \begin{bmatrix} R & P \\ P^\top & S \end{bmatrix} := \frac{\mathbf{S} + \mathbf{S}^\top}{2}.$$

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

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Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Require $\{\lambda_j^0, \mathbf{r}_j^0, \mathbf{w}_j^0\}$, $\{\mu_i^0, \mathbf{l}_i^{0\top}, \mathbf{v}_i^{0\top}\}$, D, objective order r.

- Shift data with D_s as $\mathbf{w}_j \leftarrow \mathbf{w}_j^0 + D_s$ and $\mathbf{v}_i \leftarrow \mathbf{v}_i^0 + D_s$
- Construct the r-th order Loewner interpolant $\hat{\mathbf{\Sigma}} := (-\mathbb{L}, -\mathbb{M}, \mathbf{V}, \mathbf{W}, 0_m)$
- Compute the equivalent formulation $\hat{\mathbf{\Sigma}} := (I_r, \hat{A}, \hat{B}, \hat{C}, D_s)$ with transfer $\hat{\mathbf{H}}$
- Compute projection $\hat{\mathbf{\Sigma}} \leftarrow P_\infty(\hat{\mathbf{\Sigma}})$
- Compute spectral zeros and directions $(\xi_j, \mathbf{x}_j)$ of $\hat{\mathbf{\Sigma}}$
- Set $\lambda_j \leftarrow \xi_j$, $\mathbf{r}_j \leftarrow \mathbf{x}_j$, $\mathbf{w}_j = \hat{\mathbf{H}}(\lambda_j)\mathbf{r}_j$
- Construct $\mathbb{L}$ and $\mathbb{M}$
- Set $\mathbb{M} \leftarrow \mathbb{M} - \mathbf{L}D\mathbf{R}$, $\mathbf{V} \leftarrow \mathbf{V} - \mathbf{L}D$ and $\mathbf{W} \leftarrow \mathbf{W} - D\mathbf{R}$
- Compute Chloesky decomposition $\mathbb{L} = T^\top T \Rightarrow$ Numerical issue if NSP
- Construct $\hat{\mathbf{\Sigma}}_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, D_s)$
- Construct $\hat{\mathbf{\Sigma}}_{\text{pH}} := (M, Q, J, R, G, P, N, S)$
- Set $S \leftarrow S - D_s$

Ensure $\hat{\mathbf{\Sigma}}_{\text{pH}}$ ensuring interpolatory conditions and strictly passive.

P. Benner, P. Goyal and P. Van-Dooren, *"Identification of Port-Hamiltonian Systems from Frequency Response Data"*, Systems & Control Letters, vol. 143, 2020.

Loewner extensions (passive & pH)

Passive & port-Hamiltonian

Require $\{\lambda_j^0, \mathbf{r}_j^0, \mathbf{w}_j^0\}$, $\{\mu_i^0, \mathbf{l}_i^{0\top}, \mathbf{v}_i^{0\top}\}$, D, objective order r and shift D_s.

- \+ Shift data with D_s as $\mathbf{w}_j \leftarrow \mathbf{w}_j^0 + D_s$ and $\mathbf{v}_i \leftarrow \mathbf{v}_i^0 + D_s$
- Construct the r-th order Loewner interpolant $\hat{\mathbf{\Sigma}} := (-\mathbb{L}, -\mathbb{M}, \mathbf{V}, \mathbf{W}, 0_m)$
- Compute the equivalent formulation $\hat{\mathbf{\Sigma}} := (I_r, \hat{A}, \hat{B}, \hat{C}, D_s)$ with transfer $\hat{\mathbf{H}}$
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- Construct $\mathbb{L}$ and $\mathbb{M}$
- Set $\mathbb{M} \leftarrow \mathbb{M} - \mathbf{L}(D + D_s)\mathbf{R}$, $\mathbf{V} \leftarrow \mathbf{V} - \mathbf{L}(D + D_s)$ and $\mathbf{W} \leftarrow \mathbf{W} - (D + D_s)\mathbf{R}$
- Compute Chloesky decomposition $\mathbb{L} = T^\top T$
- Construct $\hat{\mathbf{\Sigma}}_{\text{n-pH}} := (I_n, T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, D_s)$
- Construct $\hat{\mathbf{\Sigma}}_{\text{pH}} := (M, Q, J, R, G, P, N, S)$
- \+ Set $S \leftarrow S - D_s$

Ensure $\hat{\mathbf{\Sigma}}_{\text{pH}}$ ensuring interpolatory conditions and passive.

C.P-V., D. Matignon, G. Haine and P. Vuillemin, *"Data-driven port-Hamiltonian structured identification for non-strictly passive systems"*, in Proceedings of European Control Conference (ECC), Bucharest, Romania, July 2023.

Loewner extensions (passive & pH)

Passive & port-Hamiltonian (comments)

Deal with non strict passivity

This typically occurs when no direct feed-through term exist. As a consequence, the resulting spectral zeros exhibit zeros on the imaginary axis. The first and last bullets address this point.

Deal with stability

Loewner rational model $\hat{\mathbf{H}}$ may present unstable singularities. Therefore we suggest a *post stabilisation* onto $\mathcal{RH}_\infty$ as

$$P_\infty(\hat{\mathbf{H}}) = \arg \inf_{\mathbf{G}\in\mathcal{RH}_\infty} ||\hat{\mathbf{H}} - \mathbf{G}||_{\mathcal{L}_\infty}.$$

T. Breiten and B. Unger, *"Passivity preserving model reduction via spectral factorization"*, Automatica. vol. 142, pp. 110368, 2022.

M. Kohler, *"On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$"*, Linear Algebra and its Applications, vol. 443, pp. 34-49, 2014.

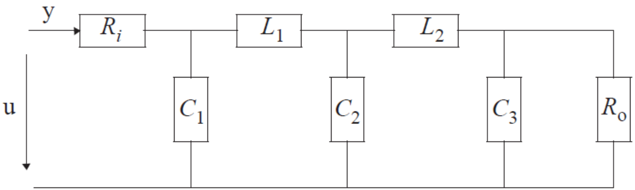
C.P-V., D. Matignon, G. Haine and P. Vuillemin, *"Data-driven port-Hamiltonian structured identification for non-strictly passive systems"*, in Proceedings of European Control Conference (ECC), Bucharest, Romania, July 2023.

Loewner extensions (passive & pH)

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)

Variables & parameters

- x_1: voltage across C_1
- x_2: current across L_1
- x_3: voltage across C_2
- x_4: current across L_2
- x_5: voltage across C_3
- u: voltage
- y: current
- $n = 5$ internal variables
- $C_i = \frac{1}{10}$ F
- $L_i = \frac{1}{10}$ H
- $R_i = \frac{1}{2}\Omega$
- $R_o = 5\Omega$



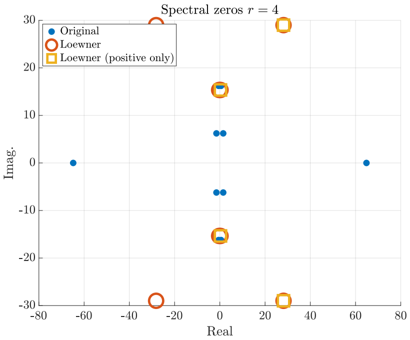
$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix}, B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

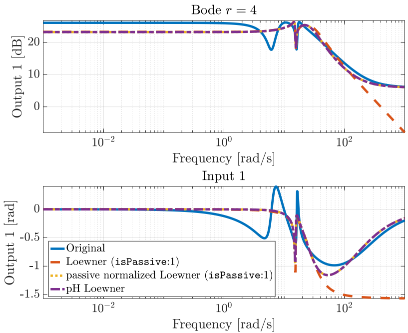
$$C = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \end{bmatrix}, D = 2$$

$$\mathbf{G}(s) = \frac{2\left(s^5 + 222\,s^4 + 840\,s^3 + 66600\,s^2 + 118000\,s + 2220000\right)}{s^5 + 22\,s^4 + 440\,s^3 + 6600\,s^2 + 38000\,s + 220000}$$

Loewner extensions (passive & pH)

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)





Here **all models result stable and passive**; passive normalized allows pH reconstruction

Loewner extensions (passive & pH)

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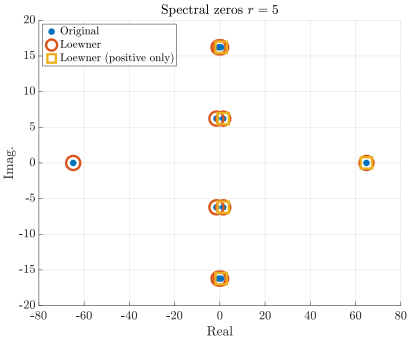

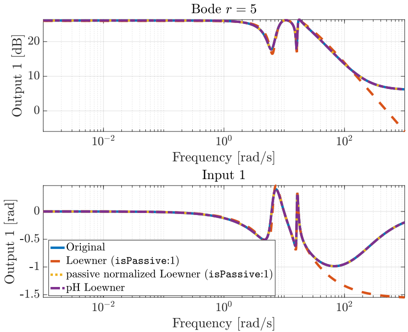

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Loewner extensions (passive & pH)

RLC ladder network (by A.C. Antoulas, CAS='siso_passive_aca')

Model

$$\begin{aligned}\dot{\mathbf{x}} &= (J-R)Q\mathbf{x} + (G-P)\mathbf{u}\\ \mathbf{y} &= (G+P)^\top Q\mathbf{x} + (N+S)\mathbf{u}\end{aligned}$$

where

$$J = \begin{pmatrix} 0 & -54.7 & -39.1 & 15.0 & 66.2\\ 54.7 & 0 & -16.3 & 0.0922 & 0.782\\ 39.1 & 16.3 & 0 & 0.0412 & 0.45\\ -15.0 & -0.0922 & -0.0412 & 0 & -6.32\\ -66.2 & -0.782 & -0.45 & 6.32 & 0 \end{pmatrix} \quad G = \begin{pmatrix} -49.6\\ -4.79\\ -5.15\\ 0.922\\ 6.23 \end{pmatrix} P = \begin{pmatrix} 45.0\\ -5.27\\ -5.65\\ 1.21\\ 8.16 \end{pmatrix}$$

$$R = \begin{pmatrix} 29.8 & 54.3 & 38.6 & -14.8 & -64.3\\ 54.3 & -1.11 & -0.482 & 0.243 & 1.44\\ 38.6 & -0.482 & -0.968 & 0.197 & 1.13\\ -14.8 & 0.243 & 0.197 & -1.59 & -0.504\\ -64.3 & 1.44 & 1.13 & -0.504 & -4.16 \end{pmatrix} \quad N = 0 \quad S = 2 \quad Q = I_5$$

Loewner extensions (passive & pH)

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)

- $\mathcal{W} = \mathcal{W}^\top \succeq 0$

$$\mathcal{W} = \begin{pmatrix} 29.8 & 54.3 & 38.6 & -14.8 & -64.3 & 45.0 \\ 54.3 & -1.11 & -0.482 & 0.243 & 1.44 & -5.27 \\ 38.6 & -0.482 & -0.968 & 0.197 & 1.13 & -5.65 \\ -14.8 & 0.243 & 0.197 & -1.59 & -0.504 & 1.21 \\ -64.3 & 1.44 & 1.13 & -0.504 & -4.16 & 8.16 \\ 45.0 & -5.27 & -5.65 & 1.21 & 8.16 & 2.0 \end{pmatrix}$$

- $\mathcal{V} = -\mathcal{V}^\top$,

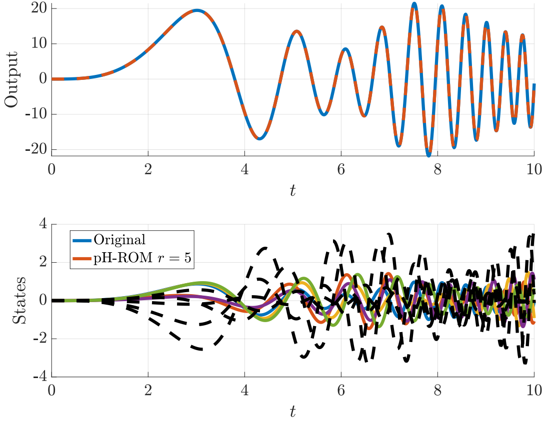
$$\mathcal{V} = \begin{pmatrix} 0 & 54.7 & 39.1 & -15.0 & -66.2 & 49.6 \\ -54.7 & 0 & 16.3 & -0.0922 & -0.782 & 4.79 \\ -39.1 & -16.3 & 0 & -0.0412 & -0.45 & 5.15 \\ 15.0 & 0.0922 & 0.0412 & 0 & 6.32 & -0.922 \\ 66.2 & 0.782 & 0.45 & -6.32 & 0 & -6.23 \\ -49.6 & -4.79 & -5.15 & 0.922 & 6.23 & 0 \end{pmatrix}$$

- $Q = Q^\top \succeq 0$

$$\Lambda(Q) = 1$$

Loewner extensions (passive & pH)

RLC ladder network (by A.C. Antoulas, `CAS='siso_passive_aca'`)



Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{array}{rcl} \mathcal{H}(t) & = & \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) \\ \mathcal{D}(t) & = & \hat{\mathbf{x}}(t)^\top (J-R)\hat{\mathbf{x}}(t) \\ \mathcal{E}(t) & = & \mathbf{u}(t)^\top \hat{\mathbf{y}}(t) \end{array}$$

Loewner extensions (passive & pH)

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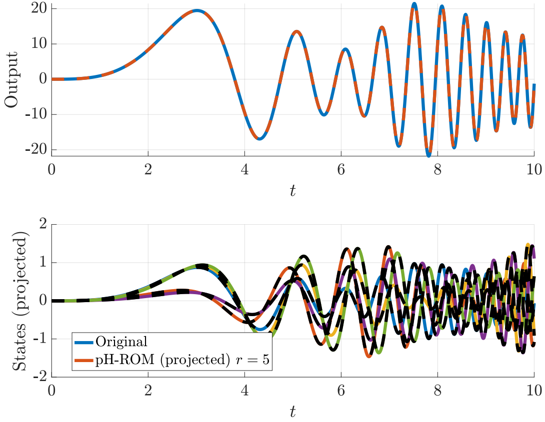

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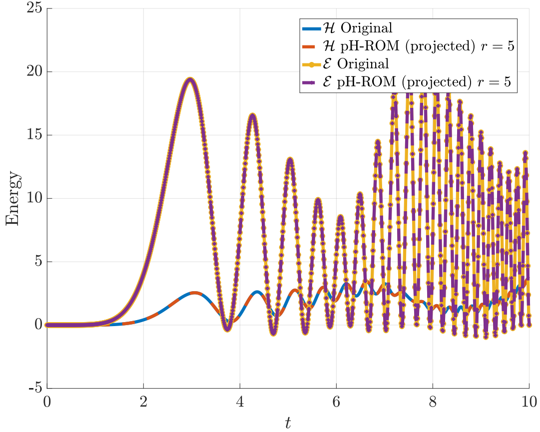

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\mathcal{E}(t) &= \mathbf{u}(t)^\top \hat{\mathbf{y}}(t)
\end{aligned}$$

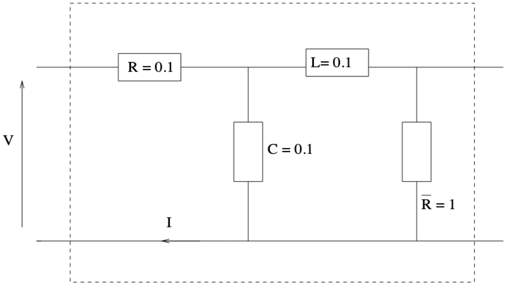
Loewner extensions (passive & pH)

RLC ladder network (by S. Gugercin, `CAS='siso_passive_gugercin'`)

Variables & parameters

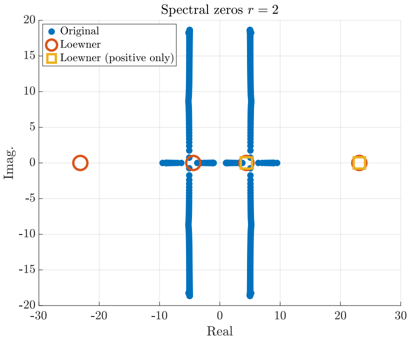
Circuit of 100 sections interconnected in cascade; each section is as shown in Figure. The input is the voltage V and the output is the current I, of the first section.

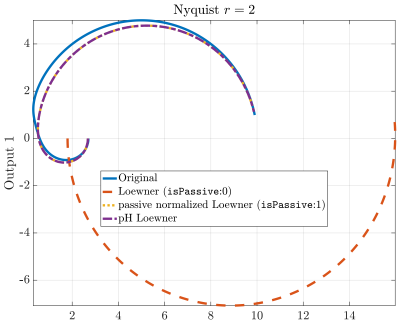
- x_i: voltage along C_i and current along L_i
- u: voltage V
- y: current I
- $n = 200$ internal variables
- $C_i = \frac{1}{10}$ F
- $L_i = \frac{1}{10}$ H
- $R_i = \frac{1}{10}\Omega$
- $\overline{R} = 1\Omega$



Loewner extensions (passive & pH)

RLC ladder network (by S. Gugercin, `CAS='siso_passive_gugercin'`)





Here **NOT all models result stable and passive**; passive normalized allows pH reconstruction

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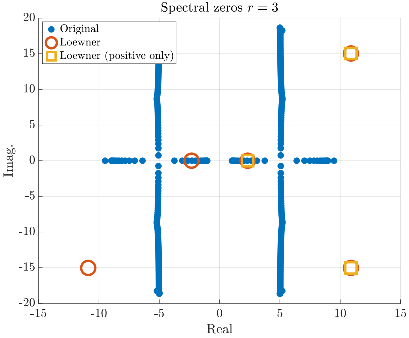

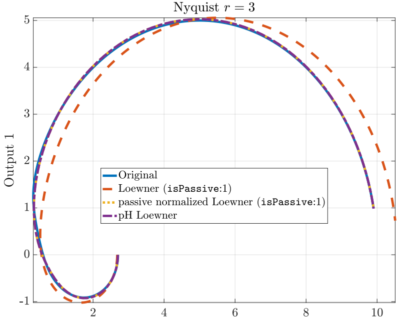

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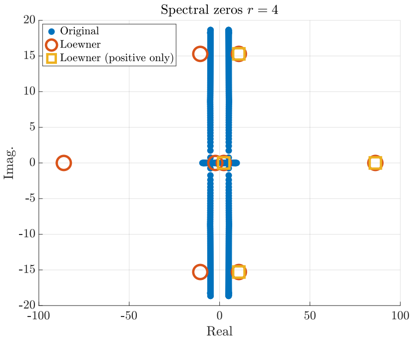

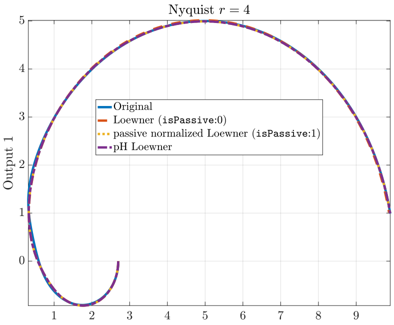

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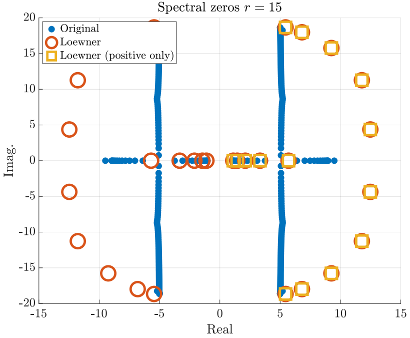

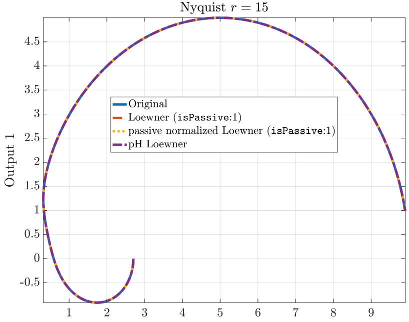

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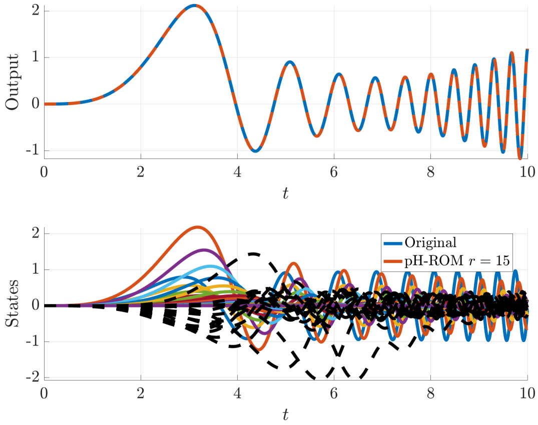

Time-domain lifting

Simulate and store $\hat{\mathbf{x}}(t)$, then lift thanks to generalized reachable subspace

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t).$$

As the system is a normalized passive one, the Hamiltonian energy $\mathcal{H}(t)$, internal dissipated power $\mathcal{D}(t)$ exchange power $\mathcal{E}(t)$ are given by

$$\begin{array}{rcl} \mathcal{H}(t) & = & \hat{\mathbf{x}}(t)^\top \hat{\mathbf{x}}(t) \\ \mathcal{D}(t) & = & \hat{\mathbf{x}}(t)^\top (J-R)\hat{\mathbf{x}}(t) \\ \mathcal{E}(t) & = & \mathbf{u}(t)^\top \hat{\mathbf{y}}(t) \end{array}$$

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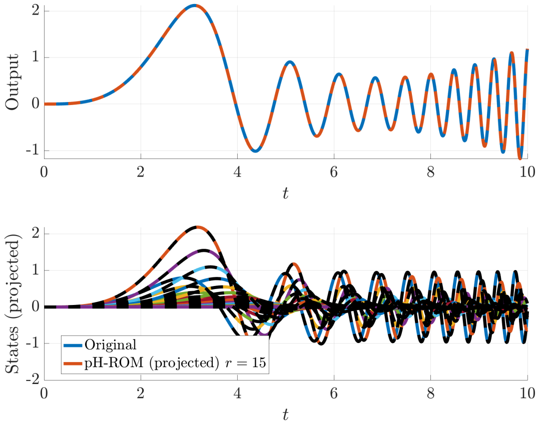

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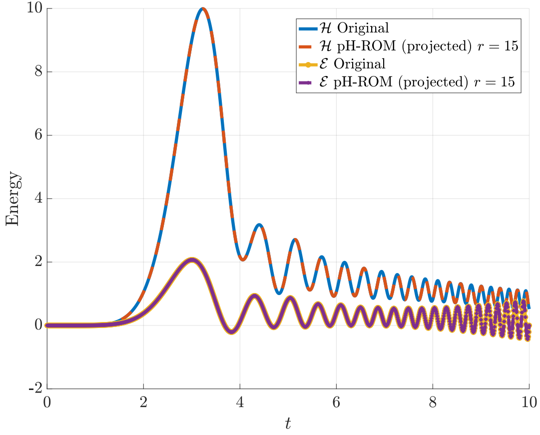

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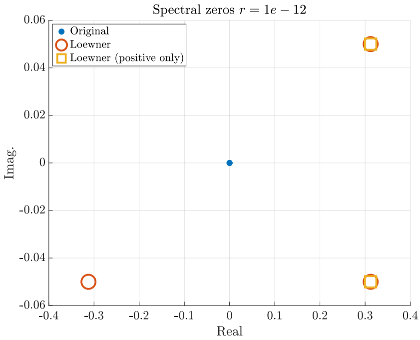
Loewner extensions (passive & pH)

Mass-spring-damper (by D. Matignon, see #1 class)



State-space form

Mass–Spring–Damper

Let us come back to the model given in the motivation.
A usual way to rewrite it in first order form is to define a vector "position–velocity". This leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t),$$

pH form ("first choice")

Finally, the pHs realisation with this choice is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \left(\begin{bmatrix} 0 & \frac{1}{m} \\ -\frac{1}{m} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \frac{b}{m^2} \end{bmatrix} \right) \begin{pmatrix} kx(t) \\ m\dot{x}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t).$$

In this case, the control F, as it is divided by m, is nothing but an acceleration, and the output is $y(t) = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} \begin{pmatrix} kx(t) \\ m\dot{x}(t) \end{pmatrix} = m\dot{x}(t)$ is the linear momentum.

- Spectral zeros very close to axis
- Numerical issues circumvented by **shifting**
- To validate...

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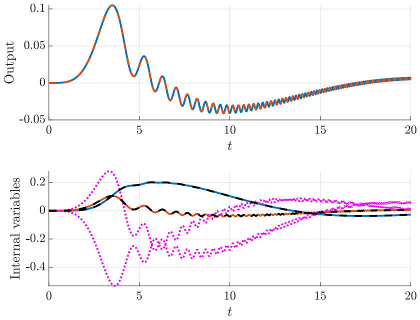

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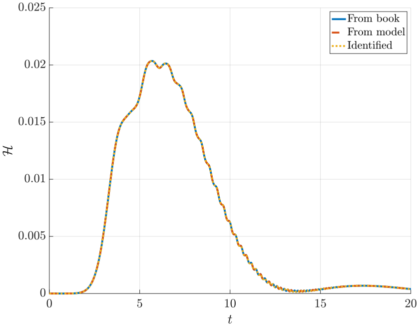

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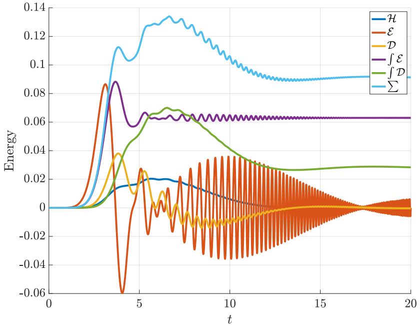

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Loewner extensions (passive & pH)

Infinite dimensional MIMO case (by A. Brugnoli reshaped by D. Matignon)

Presentation (cas 1, 2, 3 en page 5 de la section 3)

$$\frac{1}{c^2}\partial^2_{tt}w + \epsilon.\partial_t w - \partial^2_{xx}w = 0$$

Loewner extensions (passive & pH)

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Details to come

Content

Conclusions

Loewner... a versatile tool

- solves the LTI realization problem
- solves data-driven model reduction
- solves data-driven model approximation
- ... and pH
- ... and also, parametric, bilinear, quadratic...

$\rightarrow$ direct impact in engineers life

... still so much to do

- Technical references, slides and code at
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- MOR Digital Systems numerical suite
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Loewner Framework for data-driven reduced order modeling (with pH-structure preserving)

... a bridge between realization, approximation and identification

C. Poussot-Vassal
October 18, 2025
coll. with P. Vuillemin, D. Matignon, G. Haine & M. Fournié

Loewner

- A.C. Antoulas, S. Lefteriu and A.C. Ionita, "A tutorial introduction to the Loewner framework for model reduction". In: Model Reduction and Approximation, chap. 8, pp. 335-376. SIAM (2017).
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- P. Benner, P. Goyal and P. Van Dooren, "Identification of port-Hamiltonian systems from frequency response data", Systems & Control Letters, Vol. 143, p. 104741 (2020).
- A.C. Antoulas, "A new result on passivity preserving model reduction", Systems & Control Letters, vol. 54, (2005).

Conclusions

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- M. Gouzien, C. Poussot-Vassal, G. Haine and D. Matignon, "Port-Hamiltonian reduced order modelling of the 2D Maxwell equations", in journal for Computation and Mathematics in Electrical and Electronic Engineering (2020).
- J. Toledo-Zucco, D. Matignon, C. Poussot-Vassal and Y. Le Gorrec, "Structure-preserving discretization and model order reduction of boundary-controlled 1D port-Hamiltonian systems", in Systems & Control Letters, Vol. 194, December 2024, 105947.

Loewner & Loewner parametric

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- T. Vojkovic, D. Quero, C. Poussot-Vassal and P. Vuillemin, "Low-order parametric state-space modeling of MIMO systems in the Loewner framework", in SIAM Journal on Applied Dynamical Systems, Vol. 22(4), November 2023, pp. 3130-3164.
- I. V. Gosea, C. Poussot-Vassal and A. C. Antoulas, "Data-driven modeling and control of large-scale dynamical systems in the Loewner framework", Handbook in Numerical Analysis, Vol. 23, January 2022, pp. 499-530.

Conclusions

The map of mathematics (by Dominic Walliman)