From reversible to irreversible thermodynamic formulations : Control design.

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Joint work with Bernhard Maschke and Luis Mora.

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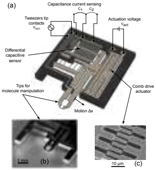
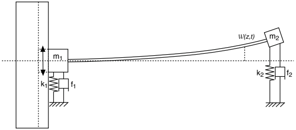
Outline

Context / Motivation : control of flexible structures

- Boundary controlled systems (e.g. Control of nanotweezers - Coll. LIMMS, Tokyo)





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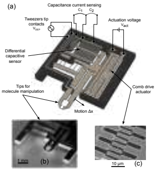
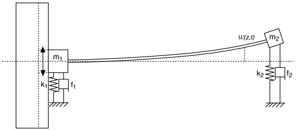
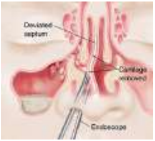
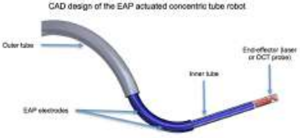




- In-domain control of distributed parameter systems (e.g. Control of smart endoscopes, FEMTO-ST)





  - Exploration, imaging, diagnosis.
  - Mini invasive surgery.
  - Toward miniaturized and *smart* endoscopes.

Context / Motivation : Toward complex systems and structures

- Soft robotics (FEMTO-ST France)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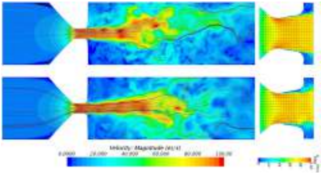
  - Modeling and control of interglotal air flows (coll. USM Chile)



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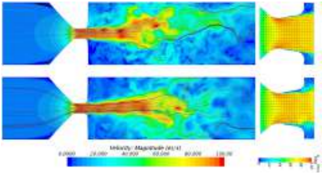

- Artificial aorta for blood pressure control (coll. EPFL Swizerland)

Context : port Hamiltonian systems

Port Hamiltonian systems (PHS)

Class of non linear dynamic systems derived from an extension to open physical systems (1992) of Hamiltonian and Gradient systems. This class has been generalized (2001) to distributed parameter systems.

$$x(t) : \begin{cases} \dot{x} = (J(x) - R(x)) \frac{\partial H(x)}{\partial x} + B(x)u \\ y = B(x)^T \frac{\partial H(x)}{\partial x} \\ \frac{dH}{dt} \leq y^T u \end{cases} \qquad x(t,\zeta) : \begin{cases} \dot{x} = (\mathcal{J}(x) - \mathcal{R}(x)) \frac{\delta H(x)}{\delta x} + \mathcal{B}_d u_d \\ y_d = \mathcal{B}_d^* \frac{\delta H(x)}{\delta x} \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\delta H(x)}{\delta x}|_\partial , \\ \frac{dH}{dt} \leq y_d^T u_d + f_\partial^T e_\partial \end{cases} ,$$

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- "Easy" to extend to non linear or systems defined on higher dimensional spaces.
- Physical properties can be efficiently used for control design (stabilization but not only).

Context : port Hamiltonian systems

In the linear 1D case this formalism has been used for

- Proving existence of solution using the semi-group theory [Le Gorrec et al., 2005].
- Stability analysis (when interconnected with linear or non linear ODEs) : asymptotic or exponential [Ramirez et al., 2017, Augner, 2016].
- Simulation through structure preserving schemes [Trenchant et al., 2018, Kotyczka et al., 2019].
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Some extensions have been proposed for

- Multidimensional systems [Skrepek, 2021].
- Implicit systems [Heidari and Zwart, 2022].
- Non linear PDE systems such as 1D or 2D-3D fluids ([Mora et al., 2021]) using Irreversible port Hamiltonian Formulations ([Ramirez et al., 2022]).

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In this talk we recall some well known results on control of PHS. Extension to IPHS.

Outline

Infinite dimensional Port Hamiltonian systems (PHS)

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -R\end{bmatrix}\begin{bmatrix} \mathcal{H}_1(\zeta)x_1(\zeta,t) \\ \mathcal{H}_2(\zeta)x_2(\zeta,t)\end{bmatrix} + \begin{bmatrix} 0 \\ I\end{bmatrix} u_d(\zeta,t) \quad (1)$$

$$y_d(\zeta,t) = \begin{bmatrix} 0 & I\end{bmatrix}\begin{bmatrix} \mathcal{H}_1(\zeta)x_1(\zeta,t) \\ \mathcal{H}_2(\zeta)x_2(\zeta,t)\end{bmatrix} \quad (2)$$

$$u_\partial = \mathcal{B}\begin{bmatrix} \mathcal{H}_1(\zeta)x_1(\zeta,t) \\ \mathcal{H}_2(\zeta)x_2(\zeta,t)\end{bmatrix},\ y_\partial = \mathcal{C}\begin{bmatrix} \mathcal{H}_1(\zeta)x_1(\zeta,t) \\ \mathcal{H}_2(\zeta)x_2(\zeta,t)\end{bmatrix} \quad (3)$$

where $x = [x_1^T, x_2^T]^T \in X := L^2([a,b],\mathbb{R}^n) \times L^2([a,b],\mathbb{R}^n)$, $\mathcal{H} = diag(\mathcal{H}_1,\mathcal{H}_2)$ and $\mathcal{H}(\zeta) = \mathcal{H}^T(\zeta)$ and $\mathcal{H}(\zeta) \geq \eta$ with $\eta > 0$ for all $\zeta \in [a,b]$, $R \in \mathbb{R}^{(n,n)}$, $R = R^T > 0$, $\mathcal{B}(\cdot)$ and $\mathcal{C}(\cdot)$ are some boundary input and boundary output mapping operators. Furthermore

$$\mathcal{G} = \sum_{i=0}^{N} G_i \frac{\partial^i}{\partial \zeta^i}, \text{ and } \mathcal{G}^* = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i}{\partial \zeta^i}$$

with $G_i \in \mathbb{R}^{(n,n)}$.

Infinite dimensional Port Hamiltonian systems (PHS)

For a sake of compactness we shall use the following notation

$$P_i = \begin{bmatrix} 0 & G_i \\ (-1)^{i+1} G_i^T & 0 \end{bmatrix}, \; R_0 = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \tag{4}$$

and the formulation of (1)

$$\frac{\partial x}{\partial t}(\zeta, t) = \sum_{i=0}^{N} P_i \frac{\partial^i}{\partial \zeta^i} \left(\mathcal{H}(\zeta) x(\zeta, t)\right) - R_0 \mathcal{H}(\zeta) x(\zeta, t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_d(\zeta, t) \tag{5}$$

$$y_d(\zeta, t) = \begin{bmatrix} 0 & I \end{bmatrix} \mathcal{H}(\zeta) x(\zeta, t) \tag{6}$$

$$u_\partial = \mathcal{B}\left(\mathcal{H}(\zeta) x(\zeta, t)\right), y_\partial = \mathcal{C}\left(\mathcal{H}(\zeta) x(\zeta, t)\right) \tag{7}$$

The total energy of the system $H(x)$ is defined by

$$H(x) = \frac{1}{2} \int_a^b \left(x^T(\zeta, t) \mathcal{H}(\zeta) x(\zeta, t) \right) d\zeta$$

Boundary controlled port Hamiltonian systems

Mixed in-domain / boundary controlled port Hamiltonian systems (IDBC-PHS)

A mixed in-domain / boundary controlled port Hamiltonian system is an infinite dimensional system of the form (5-7) where

$$u_\partial = W_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{H}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix}, \text{ and } y_\partial = W_C \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{H}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix} \tag{8}$$

with

$$W_B = \left[\frac{1}{\sqrt{2}} \left(\Xi_2 + \Xi_1 P_e\right) \quad \frac{1}{\sqrt{2}} \left(\Xi_2 - \Xi_1 P_e\right) \right], \tag{9}$$

$$W_C = \left[\frac{1}{\sqrt{2}} \left(\Xi_1 + \Xi_2 P_e\right) \quad \frac{1}{\sqrt{2}} \left(\Xi_1 - \Xi_2 P_e\right) \right], \tag{10}$$

Boundary controlled port Hamiltonian systems

where

$$P_e = \begin{bmatrix} P_1 & \cdots & (-1)^{N-1}P_N \\ \vdots & \ddots & 0 \\ (-1)^{N-1}P_N & 0 & 0 \end{bmatrix} \tag{11}$$

and Ξ_1 and Ξ_2 in $\mathbb{R}^{k\times k}$ satisfy

$$\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0, \text{ and } \Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I \tag{12}$$

The energy balance associated to the system reads

$$\frac{dH}{dt} = \int_a^b y_d^T u_d d\zeta - \int_a^b \left(x_2^T(\zeta,t)\mathcal{H}_2^T(\zeta) R \mathcal{H}_2(\zeta) x_2(\zeta,t) \right) d\zeta + y_\partial^T u_\partial \tag{13}$$

$$\leq \int_a^b y_d^T u_d d\zeta + y_\partial^T u_\partial \tag{14}$$

Boundary controlled port Hamiltonian systems

Existence of solution [Le Gorrec et al., 2005]

The operator

$$\mathcal{J} = \sum_{i=0}^{N} P_i \frac{\partial^i}{\partial \zeta^i} (\mathcal{H}(\zeta)x(\zeta,t)) - R_0 \mathcal{H}(\zeta)x(\zeta,t)$$

with domain

$$D(\mathcal{J}) = \left\{ \mathcal{H} \in H^N\left(a,b;\mathbb{R}^n\right) \mid \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(b,t) \\ \mathcal{H}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{H}x)}{\partial \zeta^{N-1}}(a,t) \end{bmatrix} \in Ker W_B \right\}$$

where W_B is defined by (9) and Ξ_1 and Ξ_2 satisfy (12), generates a unitary or contraction semigroup on X. Furthermore the system (5-7) with (9-10) and (12) defines a boundary control system.

Boundary controlled port Hamiltonian systems

The general formulation (1) allows to model a large class of systems.

For example :

- The 1D wave equation where $n = 1$, $N = 1$, $G_0 = 0$, $G_1 = 1$.
- The Euler Bernouilli beam equation. In this case $n = 1$, $N = 2$, $G_0 = 0$, $G_1 = 0$, $G_2 = 1$.
- The Timoshenko beam equation. In this case $n = 2$, $N = 1$, and

$$G_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In what follows we focus on first order differential operators

The vibrating string example

The vibrating string equation is given by

$$\frac{\partial^2 \omega(\zeta,t)}{\partial t^2} = \frac{1}{\mu(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial \omega(\zeta,t)}{\partial \zeta}\right)$$

and can be recasted in a PHS form choosing $\varepsilon = \frac{\partial \omega(\zeta,t)}{\partial \zeta}$ and $p = \mu\frac{\partial \omega(\zeta,t)}{\partial t}$ as state variables.

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon \\ p\end{pmatrix} = \begin{pmatrix}0 & \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \zeta} & 0\end{pmatrix}\begin{pmatrix}T & 0 \\ 0 & \frac{1}{\mu}\end{pmatrix}\begin{pmatrix}\varepsilon \\ p\end{pmatrix}$$

which is on the form

$$\frac{\partial x}{\partial t}(\zeta,t) \quad = \quad P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}x(\zeta,t)\right)$$

with

$$P_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, R_0 = \begin{pmatrix}0 & 0 \\ 0 & D\end{pmatrix}, \mathcal{H} = \begin{pmatrix}T & 0 \\ 0 & \frac{1}{\mu}\end{pmatrix}, \mathcal{H}x(\zeta,t) = \begin{pmatrix}\sigma(\zeta,t) \\ v(\zeta,t)\end{pmatrix}$$

The vibrating string example

The boundary port variables are

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v(b) - v(a) \\ \sigma(b) - \sigma(a) \\ \sigma(b) + \sigma(a) \\ v(b) + v(a) \end{pmatrix}$$

The boundary input and output are selected as

$$u(t) = \begin{pmatrix} v(a,t) \\ \sigma(b,t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma(a,t) \\ v(b,t) \end{pmatrix} \tag{15}$$

which can be derived choosing W and $\tilde{W}$ such that :

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \tilde{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The energy balance is then :

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = y^{\mathrm{T}}(t)u(t).$$

Outline

Static feedback control

Impedance passive case

In the impedance passive case the BCS fulfills

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = u^\top(t)y(t).$$

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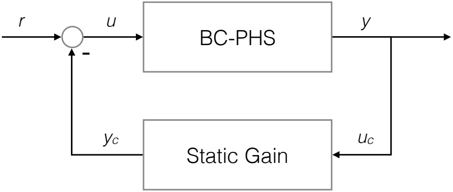


Static controller : α

- Asymptotic stability : $\alpha > 0$ + (compactness condition)
- Exponential stability [Villegas et al., 2009] : α st

$$(dE/dt) \leq -k\|(\mathcal{H}x)(t,b)\|^2_{\mathbb{R}}$$

where $k > 0$.

This result has been used for observer design [Toledo et al., 2020].

Example : Timoshenko beam

As state variables we choose

$$x_1 = \frac{\partial w}{\partial \zeta} - \phi : \quad \text{shear displacement,}$$

$$x_2 = \rho \frac{\partial w}{\partial t} : \quad \text{transverse momentum distribution,}$$

$$x_3 = \frac{\partial \phi}{\partial \zeta} : \quad \text{angular displacement,}$$

$$x_4 = I_\rho \frac{\partial \phi}{\partial t} : \quad \text{angular momentum distribution.}$$

Then the model of the beam can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_1} \frac{\partial}{\partial \zeta}\begin{pmatrix} K\, x_1 \\ \frac{1}{\rho}\, x_2 \\ EI\, x_3 \\ \frac{1}{I_\rho}\, x_4 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P_0} \underbrace{\begin{pmatrix} K\, x_1 \\ \frac{1}{\rho}\, x_2 \\ EI\, x_3 \\ \frac{1}{I_\rho}\, x_4 \end{pmatrix}}_{\mathcal{L}x}.$$

Velocity feedback

One can define the boundary port variables :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\rho^{-1}x_2)(b) - (\rho^{-1}x_2)(a) \\ (Kx_1)(b) - (Kx_1)(a) \\ (I_\rho^{-1}x_4)(b) - (I_\rho^{-1}x_4)(a) \\ (EIx_3)(b) - (EIx_3)(a) \\ (Kx_1)(b) + (Kx_1)(a) \\ (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ (EIx_3)(b) + (EIx_3)(a) \\ (I_\rho^{-1}x_4)(b) + (I_\rho^{-1}x_4)(a) \end{pmatrix}. \tag{16}$$

Let us consider stabilization by applying velocity feedback *i.e.* following BC :

$$\frac{1}{\rho(a)} x_2(a) = 0, \qquad \frac{1}{I_\rho(a)} x_4(a) = 0,$$

$$K(b)x_1(b,t) = -\alpha_1 \frac{1}{\rho(b)} x_2(b,t), \quad EI(b)x_3(b,t) = -\alpha_2 \frac{1}{I_\rho(b)} x_4(b)$$

Velocity feedback

Input mapping :

$$W_{cl} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \alpha_1 & 1 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 & 0 & 1 & \alpha_2 \end{bmatrix}$$

then

$$W_{cl} \Sigma W_{cl}^T = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \geq 0$$

As output we can choose

$$y = \begin{pmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ \frac{1}{\rho(b)}x_2(b) \\ \frac{1}{I_{\rho(b)}}x_4(b) \end{pmatrix}, \quad \text{with} \quad \widetilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Velocity feedback

Then

$$P_{W,\widetilde{W}}^{-1} = \begin{bmatrix} 2\alpha & I \\ I & 0 \end{bmatrix}, P_{W,\widetilde{W}} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix}$$

Energy balance :

$$\frac{d}{dt}E(t) = \frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t)\rangle_U - \langle \alpha y(t), y(t)\rangle_{\mathbb{R}}$$

where

$$\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = \alpha_1 |(\rho^{-1}x_2)(b,t)|^2 + \alpha_2 |(I^{-1}x_4)(b,t)|^2$$

Then

$$\begin{aligned} \| (\mathcal{L}x(b)) \|_{\mathbb{R}}^2 &= |(kx_1)(b)|^2 + |(\rho^{-1}x_2)(b)|^2 + |(EIx_3)(b)|^2 + |(I_\rho^{-1}x_4)(b)|^2 \\ &= (\alpha_1^2+1)|(\rho^{-1}x_2)(b,t)|^2 + (\alpha_2^2+1)|(I_\rho^{-1}x_4)(b)|^2 \\ &\leq \kappa\langle \alpha y(t), y(t)\rangle_{\mathbb{R}} = -\kappa \tfrac{d}{dt}E(t) \end{aligned}$$

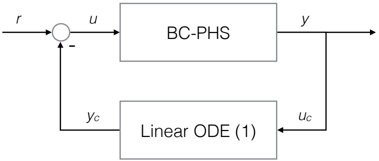
$\Rightarrow$ Stability

Dynamic boundary feedback

We consider the controller as linear finite dimensional port Hamiltonian system

$$\dot{v} = (J_c - R_c)\, Q_c v + B_c u_c, \quad y_c = B_c^T Q_c v, \quad J_c = -J_c^\top, \quad R_c = R_c^\top \geq 0$$

with storage function $E_c(t) = \frac{1}{2}\langle v(t) Q_c v(t)\rangle_{\mathbb{R}^m}$, $Q_c = Q_c^\top > 0 \in \mathbb{R}^m \times \mathbb{R}^m$.



Stability

If the following conditions are satisfied

- power preserving interconnection $u = -y_c + r$, and $u_c = y$
- the controller is assumed to be exponentially stable, i.e., $A_c := (J_c - R_c)Q_c$ is Hurwitz

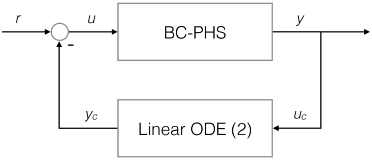
The closed loop system is asymptotically stable.

Dynamic boundary feedback

We consider the controller as linear finite dimensional port Hamiltonian system

$$\dot{v} = (J_c - R_c)\,Q_c v + B_c u_c, \quad y_c = B_c^T Q_c v + S_c u_c, \quad J_c = -J_c^\top, \quad R_c = R_c^\top \geq 0$$

with storage function $E_c(t) = \frac{1}{2}\langle v(t) Q_c v(t)\rangle_{\mathbb{R}^m}$, $Q_c = Q_c^\top > 0 \in \mathbb{R}^m \times \mathbb{R}^m$.



Stability

If the following conditions are satisfied

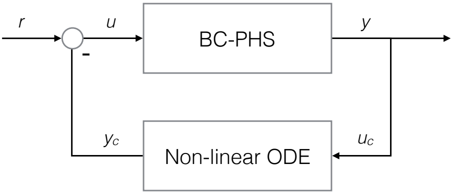
- $\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{H}x(t,b)\|^2, \ \epsilon > 0$
- power preserving interconnection $u = -y_c + r$, and $u_c = y$
- the controller is assumed to be exponentially stable, i.e., $A_c := (J_c - R_c)Q_c$ is Hurwitz and strictly input passive i.e., $S_c > 0$.

The closed loop system is exponentially stable.

This result has been used for robust tracking control design using internal model principle [Paunonen et al., 2021].

Non linear case

The previous results have been generalized to the non-linear case [Ramirez et al., 2017] (under some assumptions).



with

$$NL \left\{ \begin{array}{ll} \dot{v}_1 & = K_2 v_2 \\ \dot{v}_2 & = -\frac{\partial \mathcal{P}}{\partial v_1}(v_1)^\top - R(K_2 v_2) + B_c u_c \\ y_c & = B_c^\top K_2 v_2 + S_c u_c \end{array} \right. \tag{17}$$

where $v_1 \in \mathbb{R}^{n_c}$, $v_2 \in \mathbb{R}^{n_c}$, form the components of the state vector, $B_c \in M_{k,n_c}(\mathbb{R})$, $K_2 \in M_{n_c}(\mathbb{R})$, $K_2 = K_2^\top$, $K_2 > 0$, $S_c \in M_k(\mathbb{R})$ with $S_c = S_c^\top$ and $S_c \geq 0$.

Outline

Control by interconnection

The system is interconnected with a dynamic controller in a power preserving way.

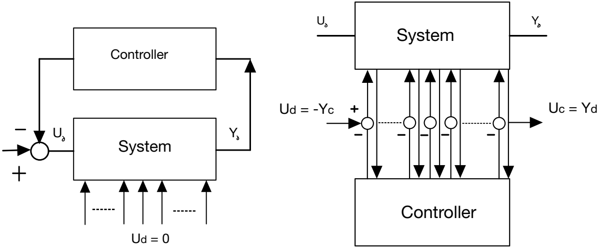


FIGURE – Control by interconnection. Boundary control (left), in domain control (right).

The closed loop energy is equal to the sum of the open loop energy and the controller energy.

Energy shaping

Objectives

Modification of the closed loop system's properties (energy shaping) + stabilization (damping injection).

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$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$

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$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$

We first look for structural invariants $C(x, x_c)$ *i.e.* $\frac{dC}{dt} = 0$

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function.

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We first look for structural invariants $C(x, x_c)$ *i.e.* $\frac{dC}{dt} = 0$

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function. In this case the closed loop energy function reads

$$H_{cl}(x, x_c) = H_{cl}(x) = H(x) + H_c(\kappa - F(x))$$

Asymptotic stability of the closed loop system in x^* is achieved using damping injection such that

$$\frac{dH_{cl}}{dt} < 0, \forall x \neq x^*.$$

Energy shaping

- Boundary control case : Asymptotic stabilisation [Macchelli et al., 2017a], Exponential stabilisation [Macchelli et al., 2020]
  We consider a dynamic controller of the form

$$\begin{cases} \dot{x}_C = (J_C - R_C)\, Q_C x_C + (G_C - P_C)\, u_C \\ y_C = (G_C + P_C)^T\, Q_C x_C + (M_C + S_C)\, u_C \end{cases} \tag{18}$$

where $x_C \in \mathbb{R}^{n_C}$ and $u_C,\ y_C \in \mathbb{R}^n$, while $J_C = -J_C^T$, $M_C = -M_C^T$, $R_C = R_C^T$, and $S_C = S_C^T$, with this further condition satisfied :

$$\begin{pmatrix} R_C & P_C \\ P_C^T & S_C \end{pmatrix} \geq 0\,. \tag{19}$$

Interconnected to the boundary of the system

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C \\ y_C \end{pmatrix} + \begin{pmatrix} u' \\ 0 \end{pmatrix}, \tag{20}$$

where $u' \in \mathbb{R}^n$ is an additional control input.

Energy shaping

Theorem

Let the open-loop BCS satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{L}} = u(t)y(t)$ and consider the previous passive finite dimensional port Hamiltonian system. Then the power preserving feedback interconnection
$u = r - y_c, y = u_c$
with $r \in \mathbb{R}^n$ the new input of the system is a BCS on the extended state space $\tilde{x} \in \tilde{X} = X \times V$ with inner product $\langle \tilde{x}_1, \tilde{x}_2 \rangle_{\tilde{X}} = \langle x_1, x_2 \rangle_{\mathcal{L}} + \langle v_1, Q_c v_2 \rangle_V$. Furthermore, the operator $\mathcal{A}_e$ defined by

$$\mathcal{A}_e \tilde{x} = \begin{bmatrix} \mathcal{J}\mathcal{L} & 0 \\ B_c\mathcal{C} & A_c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix},$$

$$D(\mathcal{A}_e) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ V \end{bmatrix} \middle| \mathcal{L}x \in H^N(a, b; \mathbb{R}^n), \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \\ v \end{bmatrix} \in \ker \tilde{W}_D \right\}$$

where $$\tilde{W}_D = \begin{bmatrix} (W + D_c\tilde{W} & C_c) \end{bmatrix}$$

generates a contraction semigroup on $\tilde{X}$.

Energy shaping

Casimir functions

Consider the closed loop boundary control system with $u' = 0$ then,

$$C(x(t), x_c(t)) = \Gamma^T x_c(t) + \int_a^b \psi^T(\zeta) x(t,\zeta) dz$$

is a Casimir function for this system if and only if $\psi \in H^1(a, b; \mathbb{R}^n)$,

$$P_1 \frac{d\psi}{dz}(\zeta) + (P_0 + G_0)\psi(\zeta) = 0 \tag{21}$$

$$(J_C + R_C)\Gamma + (G_C + P_C)\tilde{W}R \begin{pmatrix} \psi(b) \\ \psi(a) \end{pmatrix} = 0 \tag{22}$$

$$(G_C - P_C)^T\Gamma + \left[W + (M_C - S_C)\,\tilde{W}\right] R \begin{pmatrix} \psi(b) \\ \psi(a) \end{pmatrix} = 0 \tag{23}$$

Energy shaping

Sketch of the proof

$C(x_e(t))$ is a Casimir function if and only if $\frac{dC}{dt} = 0$ independently to the energy function,

$$\frac{dC}{dt} = \left\langle \frac{\delta C}{\delta x_e}, \frac{dx_e}{dt} \right\rangle_{L^2} \quad (24)$$

$$= \left\langle \frac{\delta C}{\delta x_e}, \mathcal{A}_e \mathcal{H}_e x_e \right\rangle_{L^2} \quad (25)$$

$$= \left\langle \mathcal{A}_e^* \frac{\delta C}{\delta x_e}, \mathcal{H}_e x_e \right\rangle_{L^2} + BC \quad (26)$$

$$(27)$$

Energy shaping

Proposition

Under the hypothesis that the Casimir functions exist, the closed-loop dynamics (when $u = y_c + u'$) is given by :

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1 \frac{\partial}{\partial \zeta}\frac{\delta H_{cl}}{\delta x}(x(t))(\zeta) + (P_0 - G_0)\frac{\delta H_{cl}}{\delta x}(x(t))(\zeta)$$

$$u' = W' R \begin{pmatrix} \left(\frac{\delta H_{cl}}{\delta x}(x)\right)(b) \\ \left(\frac{\delta H_{cl}}{\delta x}(x)\right)(a) \end{pmatrix} \tag{28}$$

in which δ denotes the variational derivative, while

$$H_{cl}(x(t)) = \frac{1}{2}\|x(t)\|^2_{cl} + \frac{1}{2}\left(\int_a^b \hat{\Psi}^T(\zeta)x(t,\zeta)\,dz\right)^T \times$$

$$\times\, \hat{\Gamma}^{-1} Q_C \hat{\Gamma}^{-T} \int_a^b \hat{\Psi}(\zeta)^T x(t,\zeta)\,dz \tag{29}$$

and W' is a $n \times 2n$ full rank, real matrix s.t. $W'\Sigma W'^T \geq 0$.

Extension to systems with dissipation

Proposition

The feedback law $u = \beta(x) + u'$, with u' an auxiliary boundary input, maps the original system into the target dynamical system

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\zeta) &= P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_d}{\delta x}(x(t))(\zeta) + (P_0 - G_0) \frac{\delta H_d}{\delta x}(x(t))(\zeta) \\ u'(t) &= WR \begin{pmatrix} \left(\frac{\delta H_d}{\delta x}(x(t))\right)(b) \\ \left(\frac{\delta H_d}{\delta x}(x(t))\right)(a) \end{pmatrix} \end{aligned} \tag{30}$$

with $H_d(x) = H(x) + H_a(x)$, provided that

$$P_1 \frac{\partial}{\partial \zeta} \frac{\delta H_a}{\delta x}(x) + (P_0 - G_0) \frac{\delta H_a}{\delta x}(x) = 0 \tag{31}$$

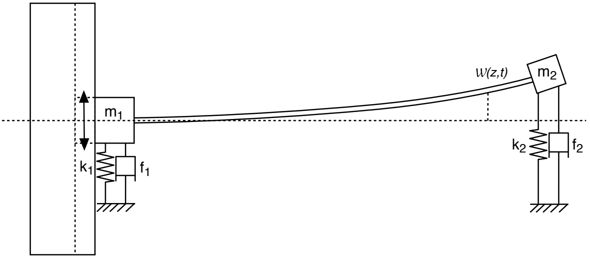
$$\beta(x) + WR \begin{pmatrix} \left(\frac{\delta H_a}{\delta x}(x)\right)(b) \\ \left(\frac{\delta H_a}{\delta x}(x)\right)(a) \end{pmatrix} = 0. \tag{32}$$

Energy shaping

With the dynamic extension or state feedback we have been able to shape a part of the closed loop energy function. It remains to prove that the closed loop system is asymptotically stable.

- We have to consider additional damping injection.
- Exponential stabilisation is not possible as "exponential stability of the controller + direct feedforward term" are necessary $\rightarrow$ no Casimir function.

Example : longitudinal (axial) vibration of a beam



State variables : deformation and linear momentum density

$$\varepsilon(t,\zeta) = \frac{\partial \varphi}{\partial \zeta}(t,\zeta), \quad p(t,\zeta) = \rho S(\zeta) v(t,\zeta) \tag{33}$$

Material's deformation is considered linear (Hooke's law) :

$$\rho S(\zeta) \frac{\partial^2 \varphi}{\partial t^2}(t,\zeta) = \frac{\partial}{\partial \zeta}\left[ES(\zeta)\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right] - D\frac{\partial \varphi}{\partial t}(t,\zeta)\mathrm{d}\zeta$$

The energy is given by (kinetic+potential) :

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left[\frac{p^2(t,\zeta)}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2(t,\zeta)\right]\mathrm{d}\zeta$$

Example : longitudinal (axial) vibration of a beam

From :

$$H(p(t,\zeta),\varepsilon(t,\zeta)) = \frac{1}{2}\int_0^L \left[\frac{p^2(t,\zeta)}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2(t,\zeta)\right] \mathrm{d}\zeta$$

We define the co-energy variables :

$$\sigma_S(t,\zeta) = \frac{\delta H}{\delta \varepsilon}(\varepsilon(t,\zeta)) = ES(\zeta)\varepsilon(t,\zeta) = S(\zeta)\sigma(t,\zeta)$$

$$v(t,\zeta) = \frac{\delta H}{\delta p}(p(t,\zeta)) = \frac{p(t,\zeta)}{\rho S(\zeta)} = \frac{\partial \varphi}{\partial t}(t,\zeta)$$

Then :

$$\frac{\partial}{\partial t}\left(\rho S(\zeta)\frac{\partial \varphi}{\partial t}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left[ES(\zeta)\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right] - D\frac{\partial \varphi}{\partial t}(t,\zeta)$$

with

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \zeta}(t,\zeta)\right) = \frac{\partial}{\partial \zeta}\left(\frac{\partial \varphi}{\partial t}(t,\zeta)\right)$$

Example : longitudinal (axial) vibration of a beam

The port-Hamiltonian formulation of the system is then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix} = \begin{pmatrix}0 & \frac{\partial}{\partial\zeta}\\ \frac{\partial}{\partial\zeta} & -D\end{pmatrix}\begin{pmatrix}ES(\zeta) & 0\\ 0 & \frac{1}{\rho S(\zeta)}\end{pmatrix}\begin{pmatrix}\varepsilon(t,\zeta)\\ p(t,\zeta)\end{pmatrix}$$

which is in the form :

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right) + (P_0 - G_0)\mathcal{H}(\zeta)x(t,\zeta) \qquad (34)$$

with $P_0 = 0$ and

$$P_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad G_0 = \begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix} \qquad \mathcal{H}(\zeta) = \begin{pmatrix}ES(\zeta) & 0\\ 0 & \frac{1}{\rho S(\zeta)}\end{pmatrix}$$

Input and output

The boundary port variables are

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v(L) - v(0) \\ \sigma_S(L) - \sigma_S(0) \\ \sigma_S(L) + \sigma_S(0) \\ v(L) + v(0) \end{pmatrix}$$

The boundary input and output are selected as

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma_S(t,L) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma_S(t,0) \\ v(t,L) \end{pmatrix} \tag{35}$$

which can be derived choosing W and $\tilde{W}$ such that :

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \tilde{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The energy balance is then :

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = -\int_0^L Dv^2(t,\zeta)\,\mathrm{d}\zeta + y^{\mathrm{T}}(t)u(t) \leq y^{\mathrm{T}}(t)u(t).$$

Lossless case : Approach based on structural invariants

We consider a dynamic controller with $n_C = 2$, $R_C = P_C = M_C = S_C = 0$, $G_C = I$ and

$$J_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

which implies that the closed-loop system is characterized by the following Casimir functions :

$$C_1(\xi_1(t), \varepsilon(t,\cdot)) = \xi_1(t) - \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta$$
$$C_2(\xi_2(t), p(t,\cdot)) = \xi_2(t) - \int_0^L p(t,\zeta)\,\mathrm{d}\zeta.$$

The controller Hamiltonian is chosen such that

$$\hat{H}_c(\xi_1, \xi_2) = \frac{1}{2}\Xi_1\xi_1^2 + \frac{1}{2}\Xi_2\xi_2^2 \tag{36}$$

Approach based on structural invariants

The closed loop energy function is :

$$H_{cl}(\varepsilon, p) = \frac{1}{2}\int_0^L \left[\frac{p^2}{\rho S(\zeta)} + ES(\zeta)\varepsilon^2\right] \mathrm{d}\zeta + \\ + \frac{1}{2}\Xi_1 \left(\int_0^L \varepsilon \, \mathrm{d}\zeta\right)^2 + \frac{1}{2}\Xi_2 \left(\int_0^L p \, \mathrm{d}\zeta\right)^2 \quad (37)$$

and the control is of the form

$$u = -y_c = -G_c \delta H_c = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L p \, \mathrm{d}\zeta \\ \int_0^L \varepsilon \, \mathrm{d}\zeta \end{pmatrix}$$

System with dissipation

Due to the dissipation $D \neq 0$, the energy-Casimir method cannot be applied. The closed loop energy function cannot be shaped in the p coordinate.

Admissible H_a :

$$\hat{H}_a(\xi_1, \xi_2) = \frac{1}{2}\Xi_1\xi_1^2 + \frac{1}{2}\Xi_2\xi_2^2$$

with

$$\begin{aligned} \xi_1(\varepsilon(t,\cdot)) &= \int_0^L \varepsilon(t,\zeta)\,\mathrm{d}\zeta \\ \xi_1(\varepsilon(t,\cdot), p(t,\cdot)) &= \int_0^L [D(L-z)\varepsilon(t,\zeta) + p(t,\zeta)]\,\mathrm{d}\zeta \end{aligned} \tag{38}$$

Leading to

$$u = -\begin{pmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{pmatrix} \begin{pmatrix} \int_0^L [D(L-z)\varepsilon(t,\zeta) + p(t,\zeta)]\,\mathrm{d}\zeta \\ \int_0^L \varepsilon\,\mathrm{d}\zeta \end{pmatrix}$$

Achievable performances

We consider now that $D = 0$, all parameters equal 1 (simulations are provided considering a finite volume approximation)

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma_S(t,L) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{u}(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} -\sigma_S(t,0) \\ v(t,L) \end{pmatrix} = \begin{pmatrix} \tilde{y}(t) \\ \bar{y}(t) \end{pmatrix}$$

and we plot the position at the end point of the system.

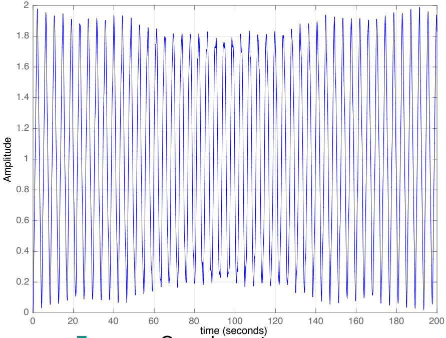


FIGURE – Open loop step response.

Simulation

We first consider the static feedback case *i.e.* when pure dissipation is added at the boundary :

$$u_2 = -k_d y_2$$

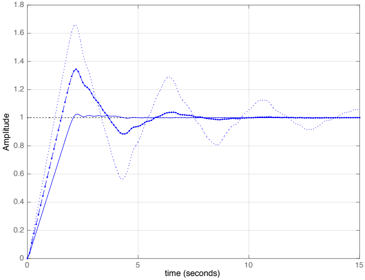


FIGURE – Step response of the closed loop system with pure dissipation term.

Simulation

In a second instance we consider the control law devoted to energy shaping in addition to a pure dissipation term :

$$u = -k_c\,(x_{22} - x_{01}) - k_d\dot{x}_{22}$$

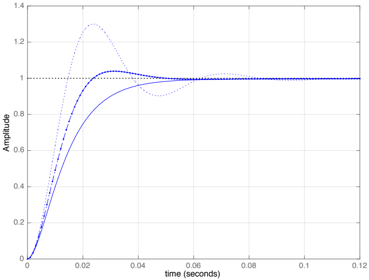
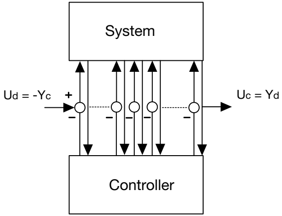


FIGURE – Step response of the closed loop system with state feedback.

Energy shaping

- In domain control case : we consider now in domain control



and the system is connected to the controller in a power preserving way :

$$\begin{pmatrix} u_d(\zeta,t) \\ y_d(\zeta,t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C(\zeta,t) \\ y_C(\zeta,t) \end{pmatrix} + \begin{pmatrix} u'(\zeta,t) \\ 0 \end{pmatrix}, \tag{39}$$

Control by interconnection : ideal case

• Ideal case : the control acts at each point ζ of the spatial domain.
The controller is of the form

$$\begin{cases} \dfrac{\partial x_C}{\partial t}(\zeta,t) = \mathcal{J}_c \mathcal{Q}_c x_C(\zeta,t) + \mathcal{B}_c u_C(\zeta,t) \\ y_C(\zeta,t) = \mathcal{B}_c{}^* \mathcal{Q}_c x_C(\zeta,t) + \mathcal{S}_c u_C(\zeta,t) \end{cases} \tag{40}$$

where $\mathcal{Q}_c(\zeta) = \mathcal{Q}_c^T(\zeta)$ and $\mathcal{Q}_c(\zeta) \geq \eta_c$ with $\eta_c > 0$ for all $\zeta \in [a,b]$, $\mathcal{S}_c$ and $\mathcal{S}_c(\zeta) = \mathcal{S}_c^T(\zeta)$ and $\mathcal{S}_c(\zeta) \geq \eta_s$ with $\eta_s > 0$ for all $\zeta \in [a,b]$ and :

$$\mathcal{B}_c = B_{c0} + B_{c1}\frac{\partial}{\partial\zeta}, \text{ and } \mathcal{J}_c = J_{c0} + J_{c1}\frac{\partial}{\partial\zeta} \tag{41}$$

with $B_{c0}, B_{c1} \in \mathbb{R}^{(n_c,1)}, J_{c0} = -J_{c0}^T, J_{c1} = J_{c1}^T \in \mathbb{R}^{(n_c,n_c)}$.

Control by interconnection : ideal case

The closed loop system reads :

$$\frac{\partial x_e}{\partial t} = \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & -(\mathcal{S}_c + R) & -\mathcal{B}_c^* \\ 0 & \mathcal{B}_c & \mathcal{J}_c \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 x_1 \\ \mathcal{H}_2 x_2 \\ \mathcal{Q}_c x_c \end{pmatrix} \tag{42}$$

Structural invariants

The closed loop system (42) admits structural invariants of the form

$$\kappa_0 = C(x_e) = \int_a^b \Psi^T x_e d\zeta \tag{43}$$

with $\Psi = (\psi_1, \psi_2, \psi_3)$ if and only if

$$-\mathcal{G}\psi_2(\zeta) = 0 = -\mathcal{B}_c\psi_2(\zeta) + \mathcal{J}_c^*\psi_3(\zeta) \tag{44}$$

$$(\mathcal{S}_c + R)\,\psi_2(\zeta) = 0 \tag{45}$$

$$\mathcal{G}\psi_1(\zeta) + \mathcal{B}_c^*\psi_3(\zeta) = 0 \tag{46}$$

$$\left.\begin{pmatrix} 0 & G_1 & 0 \\ -G_1^T & 0 & -B_{c1} \\ 0 & B_{c1}^T & J_{c1} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix}\right|_{a,b} = 0 \tag{47}$$

Energy shaping : ideal case

Energy shaping [Trenchant et al., 2017]

Choosing $\mathcal{B}_c = \mathcal{G}$ and $\mathcal{J}_c = 0$ the closed loop system (42) admits as structural invariants the function $C(x_e)$ defined by (43) and

$$\Psi = (\Psi_1, 0, \Psi_1)$$

In this case the hyperbolic system (1) connected to the dynamic controller (53) of the form

$$\begin{cases} \frac{\partial x_C}{\partial t}(\zeta, t) = \mathcal{G} u_C(\zeta, t) \\ y_C(\zeta, t) = \mathcal{G}^* \mathcal{Q}_c x_C(\zeta, t) + \mathcal{S}_c u_C(\zeta, t) \end{cases} \tag{48}$$

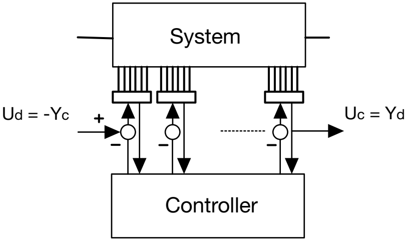
is equivalent to the system

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -(R + \mathcal{S}_c) \end{bmatrix} \begin{bmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta))\, x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{bmatrix} \tag{49}$$

$$u_\partial = \mathcal{B} \begin{bmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta))\, x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{bmatrix}, \; y_\partial = \mathcal{C} \begin{bmatrix} (\mathcal{H}_1(\zeta) + \mathcal{Q}_c(\zeta))\, x_1(\zeta, t) \\ \mathcal{H}_2(\zeta) x_2(\zeta, t) \end{bmatrix} \tag{50}$$

Control by interconnection

• Non ideal case : the distributed parameter system is actuated through piecewise constant elements.



Early lumping approach

The system is first discretized using a structure preserving method (mixed finite element method [Golo et al., 2004]) such that the approximation of (1) is again a PHS with n elements :

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = (J_n - R_n) \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + B_b u_b + \begin{pmatrix} 0 \\ B_{0d} \end{pmatrix} \mathbf{u}_d, \tag{51a}$$

$$y_b = B_b^T \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + D_b u_b, \tag{51b}$$

$$\mathbf{y}_d = \begin{pmatrix} 0 & B_{0d}^T \end{pmatrix} \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{51c}$$

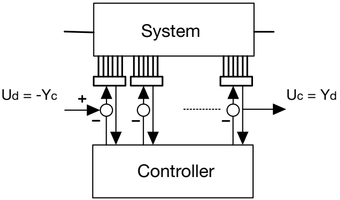
where $x_{id} = \begin{pmatrix} x_i^1 & \cdots & x_i^n \end{pmatrix}^T \in \mathbb{R}^{np \times 1}$ for $i \in \{1, \cdots, 2p\}$,

$$J_n = \begin{pmatrix} 0 & J_i \\ -J_i^T & 0 \end{pmatrix} \quad \text{and} \quad R_n = \begin{pmatrix} 0 & 0 \\ 0 & R_d \end{pmatrix},$$

The discretized energy reads :

$$H_d(x_{1d}, x_{2d}) = \frac{1}{2} \left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} \right). \tag{52}$$

Control by interconnection



The controller is designed as finite dimensional PHS of the form :

$$\begin{cases} \dot{x}_c = (J_c - R_c)\, Q_c x_c + B_c u_c, \\ y_c = B_c^T Q_c x_c + D_c u_c, \end{cases} \tag{53}$$

interconnected in a power preserving way through the relation

$$\begin{pmatrix} u_d \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} y_d \\ y_c \end{pmatrix}, \text{ where } M = \mathbb{I}_m \otimes \mathbf{1}_{k\times 1} \in \mathbb{R}^{n\times m}, \tag{54}$$

Control by interconnection

The closed loop system is given by

$$\dot{x}_{cl} = (J_{cl} - R_{cl})\, Q_{cl} x_{cl}, \tag{55}$$

where $x_{cl} = \begin{pmatrix} x_{1d}^T, & x_{2d}^T, & x_c^T \end{pmatrix}^T$, $Q_{cl} = \text{diag}\begin{pmatrix} Q_1, & Q_2, & Q_c \end{pmatrix}$,

$$J_{cl} = \begin{pmatrix} O & J_i & 0 \\ -J_i^T & 0 & -B_{0d} M B_c^T \\ 0 & B_c M^T B_{0d}^T & J_c \end{pmatrix}, \quad R_{cl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_d + B_{0d} M D_c M^T B_{0d}^T & 0 \\ 0 & 0 & R_c \end{pmatrix}.$$

The Hamiltonian of the controller (53) is :

$$H_c(x_c) = \frac{1}{2} x_c^T Q_c x_c. \tag{56}$$

Therefore, the closed loop Hamiltonian function reads :

$$H_{cld}(x_{1d}, x_{2d}, x_c) = H_d(x_{1d}, x_{2d}) + H_c(x_c). \tag{57}$$

Energy shaping

Approximate energy shaping [Liu et al., 2021]

Choosing $J_c = 0$, and $R_c = 0$, the closed loop system (55) admits :

$$C(x_{1d}, x_c) = B_c M^T B_{0d}^T J_i^{-1} x_{1d} - x_c \tag{58}$$

as structural invariant along the closed loop trajectories. The control law (54) is a PI action equivalent to the state feedback :

$$\mathbf{u}_d = -B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} x_{1d} - D_c M^T B_{0d}^T Q_2 x_{2d}. \tag{59}$$

Therefore, the closed loop system yields :

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = \begin{pmatrix} 0 & J_i \\ -J_i^T & -(R_d + B_{0d} M D_c M^T B_{0d}^T) \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{60}$$

where : $\tilde{Q}_1 = Q_1 + J_i^{-T} B_{0d} M B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1}$.

$B_c^T Q_c B_c$ can be designed to minimise $\left\| \tilde{Q}_1 - Q_m \right\|_F$ (Convex optimization problem)

Stability analysis

The controller is now connected to the infinite dimensional system leading to :

$$\dot{\mathcal{X}} = \underbrace{\begin{pmatrix} (\mathcal{J} - \mathcal{R} - \mathcal{B}D_c\mathcal{B}^*) & -\mathcal{B}B_c^T \\ B_c\mathcal{B}^* & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H} & 0 \\ 0 & Q_c \end{pmatrix}}_{\mathcal{A}_{cl}} \mathcal{X}, \tag{61}$$

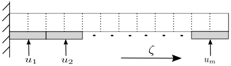
where $\mathcal{X} = \begin{pmatrix} x^T & x_c^T \end{pmatrix}^T \in X_s$ where $X_s = L_2\left([0, L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.

Existence of solution, stability analysis

- The operator $\mathcal{A}_{cl}$ defined in (61) generates a contraction semigroup on $X_s = L_2\left([0, L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.
- The operator $\mathcal{A}_{cl}$ has a compact resolvent.
- Asymptotic stability : For any $\mathcal{X}(0) \in L_2\left([0, L], \mathbb{R}^{2n}\right) \times \mathbb{R}^m$, the unique solution of (61) tends to zero asymptotically, and the closed loop system (61) is globally asymptotically stable.

Energy shaping : application (1)

We consider the control of a weakly damped Timoshenko beau using 50 homogeneously distributed patches.

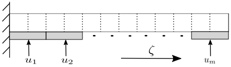


Application case (2) (damping injection)

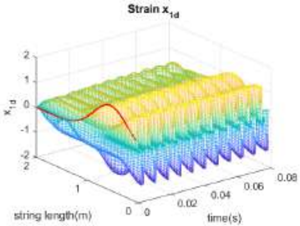
Application case (2) (energy shaping +damping injection)

Energy shaping : application

We consider the control of a weakly damped vibratring string using *m* homogeneously distributed patches (n discretization elements).



We consider the case with m patches, *i.e.* $m = 10$, $n = 50$ and $k = 5$. The initial conditions are set to a spatial distribution $x_1(\zeta, 0) = \mathcal{N}(1.5, 0.113)$ for the strain distribution and to zero for velocity distribution.



Control by interconnection

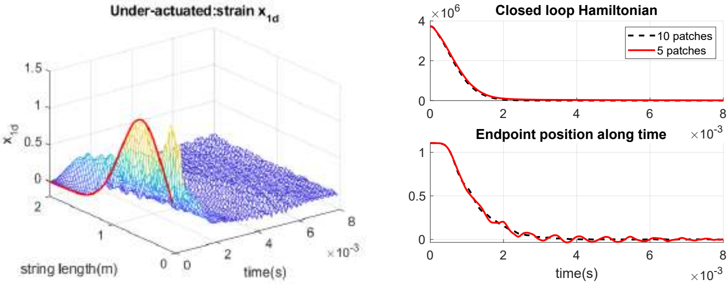


FIGURE – Closed loop evolution of the angular strain for $m = 10$ (a), Hamiltonian function and endpoint position (b) in the under-actuated case for $m = 10$, $m = 5$.

Control by interconnection

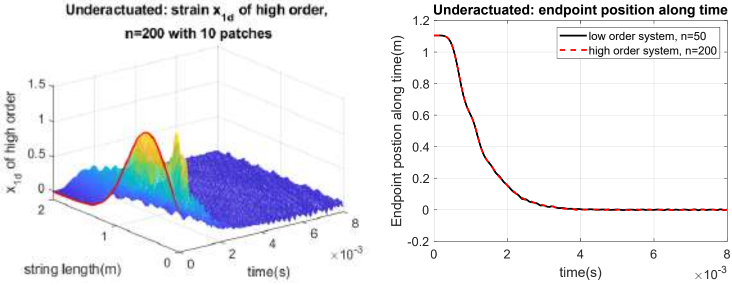


FIGURE – Closed loop evolution of the angular strain of the high order system (a), and comparison of the endpoint position of the low order and high order systems using the same controller (b).

Control by interconnection

Achievable performances

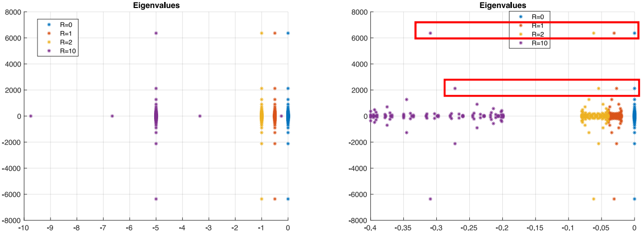


FIGURE – Control by interconnection. Full actuation (left), partial actuation (right).

Control by interconnection (Achievable performances)

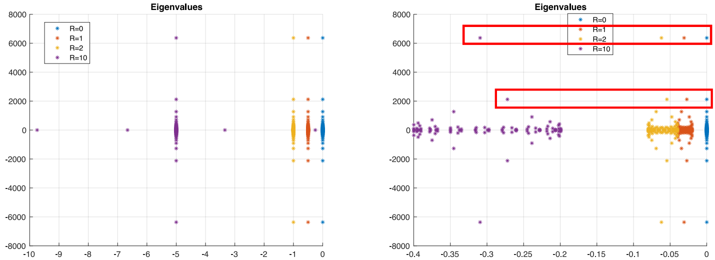


FIGURE – Control by interconnection. Full actuation (left), partial actuation (right).

Outline

Observer design

In many cases the power conjugated variable is not (completely) measurable. In this case one has to use an observer.

$$\mathcal{U}\begin{cases}\frac{\partial x}{\partial_t}(\zeta,t) = P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t),\\ W_{\mathcal{B}}\begin{pmatrix}\mathbf{f}_\partial(t)\\ \mathbf{e}_\partial(t)\end{pmatrix} = u(t), \quad x(\zeta,0) = x_0(\zeta),\\ y(t) = W_{\mathcal{C}}\begin{pmatrix}\mathbf{f}_\partial(t)\\ \mathbf{e}_\partial(t)\end{pmatrix},\\ y_m(t) = \mathcal{C}_m x(\zeta,t),\end{cases} \tag{62}$$

$$\hat{\mathcal{U}}\begin{cases}\frac{\partial\hat{x}}{\partial_t}(\zeta,t) = P_1\frac{\partial}{\partial\zeta}(\mathcal{H}\hat{x}(\zeta,t)) + P_0(\mathcal{H}\hat{x}(\zeta,t)),\\ W_{\mathcal{B}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right) = \hat{u}(t), \quad \hat{x}(\zeta,0) = \hat{x}_0(\zeta)\\ \hat{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right),\\ \hat{y}_m(t) = \mathcal{C}_m\hat{x}(\zeta,t),\end{cases} \tag{63}$$

Since the system $\hat{\mathcal{U}}$ in (63) is virtual, the input $\hat{u}(t)$ is designed with all the available information, i.e. $\hat{u}(t) = f(u(t), y_m(t), \hat{x}(\zeta,t))$, where $u(t)$ and $y_m(t)$ are considered known from (62) and $f(\cdot)$ is a function to be designed.

Observer design

Defining

$$\tilde{x}(\zeta,t) := x(\zeta,t) - \hat{x}(\zeta,t). \tag{64}$$

Then, from (62) and (63), we obtain the error dynamics equations as follows :

$$\tilde{\mathcal{U}} \begin{cases} \frac{\partial \tilde{x}}{\partial_t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_{\mathcal{B}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right) = \tilde{u}(t), \quad \tilde{x}(\zeta,0) = \tilde{x}_0(\zeta), \\ \tilde{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right). \end{cases} \tag{65}$$

We define the Hamiltonian of the error system as :

$$\tilde{H}(t) = \frac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}} = \frac{1}{2}\int_a^b \tilde{x}(\zeta,t)^T \mathcal{H}(\zeta)\tilde{x}(\zeta,t)d\zeta. \tag{66}$$

Since $W_{\mathcal{B}}$ and $W_{\mathcal{C}}$ are such that $W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^T = I$, the time derivative of $\tilde{H}(t)$ satisfies

$$\dot{\tilde{H}}(t) = \tilde{u}(t)^T \tilde{y}(t). \tag{67}$$

Observer design

Full sensing case

Consider the BC-PHS (62)with $y_m(t) = y(t)$. The state of the observer (63) with

$$\hat{u}(t) = u(t) + L(y_m(t) - \hat{y}_m(t)), \tag{68}$$

converges exponentially to the state of the BC-PHS (62) if $0 < L + L^T \in \mathbb{R}^{n\times n}$.

Partial sensing case

Consider the BC-PHS (62) with $y_m(t) = C_m y(t)$ and $C_m = \begin{pmatrix} I_p & 0_{p\times n-p} \end{pmatrix} \in \mathbb{R}^{p\times n}$, $0 < p < n$. The states of the observer (63) with

$$\hat{u}(t) = u(t) + C_m^T L(y_m(t) - \hat{y}_m(t)) \quad \text{and} \quad L \in \mathbb{R}^{p\times p} \tag{69}$$

converges exponentially to the state of the BC-PHS (62) if L is such that $C_m^T L^T C_m + C_m^T L C_m \geq 0$, and one of the following conditions is satisfied ($\gamma > 0$)

$$\begin{aligned} &\|\mathcal{H}(b)\tilde{x}(b,t)\|^2_{\mathbb{R}} \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) \quad \textit{or} \\ &\|\mathcal{H}(a)\tilde{x}(a,t)\|^2_{\mathbb{R}} \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t), \end{aligned} \tag{70}$$

Observer design

Position measurement

Consider the BC-PHS (62). Assume that the measurement is on the following form :

$$y_m(t) = \int_0^t C_m y(\tau) d\tau + y_m(0), \text{ with } C_m = \begin{pmatrix} 0_{p\times n-p} & I_p \end{pmatrix}. \quad (71)$$

Assume that the BC-PHS is approximately observable with respect to the output $C_m y(t)$. The state of the observer (63) with

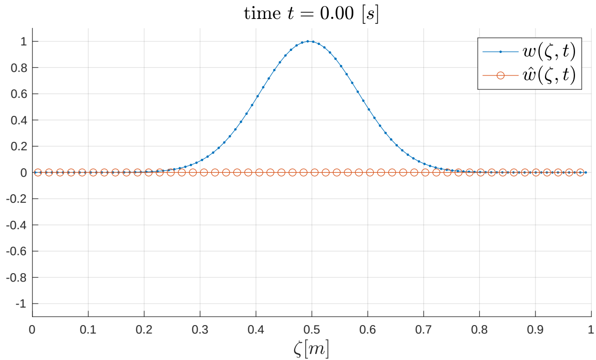
$$\begin{aligned} \hat{u}(t) &= u(t) + C_m^T L_1 (y_m(t) - \hat{y}_m(t) + \theta(t)), \\ \dot{\theta}(t) &= -L_2 (y_m(t) - \hat{y}_m(t) + \theta(t)), \quad \theta(0) = \theta_0. \end{aligned} \quad (72)$$

converges asymptotically to the state of the BC-PHS (62) if $L_1, L_2 \in \mathbb{R}^{p\times p}$ are both positive definite matrices.

Implementation on the elastic string example

We consider now

- The position of the end point *i.e.* $\omega(b, t)$, is measured .
- The state is reconstructed using a Luenberger PH finite dimensional observer (the control uses $\hat{\omega}(b, t)$ and $\hat{v}(b, t)$)$\Rightarrow$ the closed loop stability is guaranteed [Toledo et al., 2020].



Outline

Heat equation

Balance equation on u

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial z}\left(-\lambda \frac{\partial T}{\partial z}\right)$$

where λ denotes the heat conduction coefficient. From Gibbs' equation $du = Tds$ and

$$\frac{\partial s}{\partial t} = -\frac{1}{T}\frac{\partial}{\partial z}\left(-\lambda \frac{\partial T}{\partial z}\right)$$

or alternatively

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\lambda}{T}\frac{\partial T}{\partial z}\right) + \frac{\lambda}{T^2}\left(\frac{\partial T}{\partial z}\right)^2$$

One can notice that : $T = \frac{\delta U}{\delta s}$ where $U = \int_a^b u dz$.

Heat equation

IPHS formulation

$$\frac{\partial s}{\partial t} = \frac{\lambda}{T^2}\frac{\partial T}{\partial z}\frac{\partial}{\partial z}\left(\frac{\delta U}{\delta s}\right) + \frac{\partial}{\partial z}\left(\frac{\lambda}{T^2}\frac{\partial T}{\partial z}\left(\frac{\delta U}{\delta s}\right)\right)$$

which is an IPHS where $P_0 = 0$, $P_1 = 0$, $G_0 = 0$, $G_1 = 0$, $g_s = 1$ and $r_s = \gamma_s\{S|U\}$ with $\gamma_s = \frac{\lambda}{T^2}$ and $\{S|U\} = \frac{\partial T}{\partial z}$. In this case $P_e = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $n = 1$ and $m = 1$.
Choosing $\Xi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Xi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ the boundary inputs and outputs of the system are

$$v(t) = \begin{bmatrix} \left(\frac{\lambda_s}{T}\frac{\partial T}{\partial z}\right)(t,b) \\ -\left(\frac{\lambda_s}{T}\frac{\partial T}{\partial z}\right)(t,a) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} T(t,b) \\ T(t,a) \end{bmatrix},$$

respectively the entropy flux and the temperature at each boundary.

Control design

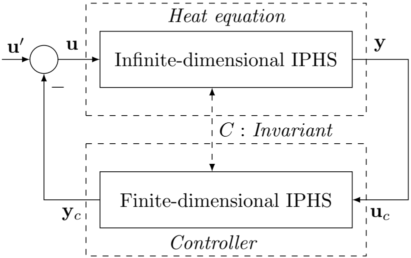


FIGURE – CbI of the heat equation

We consider reflective BC at 0 and control at L.

$$\mathbf{u}_c = \mathbf{y}, \qquad \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = -\mathbf{y}_c + \begin{bmatrix} \mathbf{u}' \\ 0 \end{bmatrix} \tag{73}$$

A nonlinear finite dimensional boundary controller is designed by extending the control by interconnection (CbI) for BC-PHS [Macchelli et al., 2017] to BC-IPHS. The objective is to characterize the conditions for the existence of closed-loop invariant functions, which are then used to shape the closed-loop energy function and assign the closed-loop entropy

Control design

The controller is looked for on the form

$$\dot{x}_c = 0e_c + G_c(x_c, \mathbf{u}_c)\mathbf{u}_c, \qquad \mathbf{y}_c = G_c^\top(x_c, \mathbf{u}_c)e_c, \tag{74}$$

where $e_c = \partial_{x_c} H_c$.
Denoting by $\mathcal{C}$ and $\mathcal{B}$ the boundary operators such that the output and input in (**??**) can be expressed as $\mathbf{y} = \mathcal{C}e_s$ and $\begin{bmatrix}\mathbf{u}\\0\end{bmatrix} = \mathcal{B}e_s$, respectively, we obtain the coupled PDE-ODE system that follows

$$\begin{aligned}
\underbrace{\begin{bmatrix}\partial_t s\\ \dot{x}_c\end{bmatrix}}_{\dot{\mathbf{x}}_{cl}} &= \underbrace{\begin{bmatrix} r_s\partial_z(\cdot) + \partial_z(r_s\cdot) & 0\\ G_c(x_c, \mathbf{u}_c)\mathcal{C} & 0\end{bmatrix}}_{\mathcal{J}_{cl}} \underbrace{\begin{bmatrix}e_s\\ e_c\end{bmatrix}}_{\mathbf{e}_{cl}}\\
\begin{bmatrix}\mathbf{u}'\\ 0\end{bmatrix} &= \underbrace{\begin{bmatrix}\mathcal{B} & G_c^\top(x_c, \mathbf{u}_c)\end{bmatrix}\mathbf{e}_{cl}}_{W_{B_{cl}}}
\end{aligned} \tag{75}$$

where $\mathbf{e}_{cl} \in \mathbb{E}_{cl}$ denote the co-states of the closed-loop system, with $\mathbb{E}_{cl} = \mathbb{T} \times \mathbb{R}$ the corresponding co-state space and the inner product

$$\left\langle \mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} = \int_0^L f_1^1(z) f_1^2(z)\mathrm{d}z + f_2^1 f_2^2$$

for all $\mathbf{f}^i = [f_1^i(z) \quad f_2^i]^\top \in \mathbb{E}_{cl}$.

Control design

The operator $\mathcal{J}_{cl}$ satisfies

$$\left\langle \mathbf{f}^1, \mathcal{J}_{cl}\mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} = \left\langle -\mathcal{J}_{cl}\mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} + [\mathcal{C}f_1^1]^\top W_{B_{cl}}\mathbf{f}^2 + [W_{B_{cl}}\mathbf{f}^1]^\top \mathcal{C}f_1^2. \tag{76}$$

Setting $W_{B_{cl}}\mathbf{f}^i = 0$, $\forall \mathbf{f}^i \in \mathbb{E}_{cl}$ we obtain that $\langle \mathbf{f}^1, \mathcal{J}_{cl}\mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \langle -\mathcal{J}_{cl}\mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}}$, i.e., $\mathcal{J}_{cl}$ is formally skew-adjoint on the space $\mathbb{E}_{cl}$.

Control design

The operator $\mathcal{J}_{cl}$ satisfies

$$\left\langle \mathbf{f}^1, \mathcal{J}_{cl}\mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} = \left\langle -\mathcal{J}_{cl}\mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} + [\mathcal{C}f_1^1]^\top W_{B_{cl}}\mathbf{f}^2 + [W_{B_{cl}}\mathbf{f}^1]^\top \mathcal{C}f_1^2. \tag{76}$$

Setting $W_{B_{cl}}\mathbf{f}^i = 0, \forall \mathbf{f}^i \in \mathbb{E}_{cl}$ we obtain that $\langle \mathbf{f}^1, \mathcal{J}_{cl}\mathbf{f}^2 \rangle_{\mathbb{E}_{cl}} = \langle -\mathcal{J}_{cl}\mathbf{f}^1, \mathbf{f}^2 \rangle_{\mathbb{E}_{cl}}$, i.e., $\mathcal{J}_{cl}$ is formally skew-adjoint on the space $\mathbb{E}_{cl}$.

Casimir invariants

Consider the boundary control system (75) with $\mathbf{u}' = 0$. A function $C : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is an invariant of (75) if $\dot{C} = 0$ along the trajectories (75) for any $\mathbf{e}_{cl}$.

Control design

Proposition

We consider $C(s, x_c)$ of the form

$$C(s, x_c) = \Gamma x_c + \int_0^L f(s(z))\mathrm{d}z = \kappa \tag{77}$$

where κ is a constant and $f(s) \in H^1([0, L], \mathbb{R})$ is a continuous function. Then (77) is an invariant if

$$\langle \mathcal{J}_{cl}\boldsymbol{\epsilon}, \mathbf{e}_{cl} \rangle = 0 \tag{78}$$

$$\begin{bmatrix} W_B & G_c^\top(x_c, \mathbf{u}_c) \end{bmatrix} \boldsymbol{\epsilon} = 0 \tag{79}$$

where

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_s \\ \epsilon_c \end{bmatrix} = \begin{bmatrix} \delta_s C \\ \partial_{x_c} C \end{bmatrix} = \begin{bmatrix} \partial_s f(s) \\ \Gamma \end{bmatrix} \tag{80}$$

Control design

Proposition

The function $C(s, x_c) = \Gamma x_c + \int_0^L f(s(z))\mathrm{d}z$ is an invariant if $f(s) = \alpha u(s) + c_1$ where c_1 is a function that does not depend on s. The state of the control system (74) is then given by the state feedback

$$x_c = -\frac{\alpha}{\Gamma}\int_0^L u(s)\mathrm{d}z + \bar{k}/\Gamma = -\frac{\alpha}{\Gamma}\mathcal{H}(s) + \bar{k}/\Gamma \tag{81}$$

with $\bar{k} = \left(k + \int\limits_0^L c_1 \mathrm{d}z\right)$ and the controller energy function is

$$H_c = \frac{\Gamma}{\alpha}x_c + k_c = -\mathcal{H}(s) + k' \tag{82}$$

where $k' = \frac{\bar{k}}{\alpha} + k_c$, with k_c a constant. Furthermore, the energy function of the closed-loop system is constant and equal to k'.

Control design

Although the energy H_{cl} of the closed-loop system is constant, the state of the controller provides a measure of the internal energy of the heat equation.

- Setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = 0$ the state of the controller is

$$x_c(t) = \int_0^L u(z,t)\mathrm{d}z = \mathcal{H}(t), \quad x_c(0) = \mathcal{H}(0)$$

  i.e., x_c is a measure of the total energy of the IPHS.

- Setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = -\mathcal{H}(0)$,

$$x_c(t) = \int_0^L (u(z,t) - u_0)\,\mathrm{d}z = \Delta\mathcal{H}, \quad x_c(0) = 0$$

  with $\Delta\mathcal{H} = \mathcal{H}(t) - \mathcal{H}(0)$. In this case x_c is a measure of the energy supplied by the controller to the process.

- Setting $\alpha = -1$, $\Gamma = \mathcal{H}(0)$ and $\bar{k} = -\mathcal{H}(0)$ the controller state characterizes the normalized total energy of the IPHS, $x_c(t) = \mathcal{H}/\mathcal{H}(0)$ if $x_c(0) = 1$, and the normalized energy supplied by the controller, $x_c(t) = \Delta\mathcal{H}/\mathcal{H}_0$ if $x_c(0) = 0$.

Control design

Proposition

The boundary controller (74) with

$$G_c^\top = \frac{\alpha}{\Gamma T|_L}\begin{bmatrix} g(x_c,\mathbf{y}) \\ 0 \end{bmatrix} \tag{83}$$

where

$$g(x_c,\mathbf{y}) = \phi_L(x_c)(T - T^*)|_L + \phi_0(x_c)(T - T^*)|_0$$

with $\phi_0(x_c) \geq 0$ and $\phi_L(x_c) > \phi_0(x_c)\left(\frac{L^2\phi_0(x_c)}{2k} - 1\right)$, exponentially stabilizes (**??**) at the desired equilibrium profile T^*.

The proof follows from considering the following closed loop Lyapunov function

$$\mathcal{V} = \int_0^L \frac{1}{2}(T - T^*)^2 \mathrm{d}z$$

where $T^* \in \mathcal{T}^*$ is the reachable dynamic equilibrium profile of the temperature.

Control design

Proposition

We consider now that the system is fully actuated. The boundary controller (74) with

$$G_c^\top = \frac{\alpha}{\Gamma}\left(\begin{bmatrix} -\frac{km^*}{T|_L} \\ -\frac{km^*}{T|_0} \end{bmatrix} + \frac{1}{c_v}\Phi(x_c)\begin{bmatrix} T(T-T^*)|_L \\ -T(T-T^*)|_0 \end{bmatrix}\right) \tag{84}$$

where $\Phi(x_c) = \Phi(x_c)^\top \geq 0$, exponentially stabilizes the closed loop system at the desired equilibrium profile T^*.

The proof follows from considering the following closed loop Lyapunov function

$$\mathcal{V} = \int_0^L \frac{1}{2}(T - T^*)^2 \mathrm{d}z$$

where $T^* \in \mathcal{T}^*$ is the reachable dynamic equilibrium profile of the temperature.

Heat equation

We consider a coper bar of length $L = 0.1m$ and a cross-sectional area of $10^{-4}m^2$.

$$T_0 = T(\xi, 0) = 200\xi + 330, \forall\xi \in [0, 0.1]$$

the desired temperature equilibrium profile is defined as

$$T^* = 325, \forall\xi \in [0, 0.1]$$

The boundary controller acts on the entropy flux at the boundary $\xi = L$,i.e., $u = -q_s|L$.
To move T_0 to T^* we use the boundary controller described in Proposition 3.

Heat equation

Control design

Using the controller (74) with $\alpha = -1$, $\Gamma = \mathcal{H}_0$ and $\bar{k} = 0$, the controller state variable represents a measurement of the normalized total energy if $x_c(0) = 1$, i.e.,

$$x_c = \frac{1}{\mathcal{H}_0} \int_0^L u(z,t) \mathrm{d}z.$$

Assuming temperature measurements at both boundaries we select

$$\phi_L(x_c) = 5x_c \quad \text{and} \quad \phi_0(x_c) = 10x_c,$$

Control law

$$\begin{aligned} \mathbf{u} &= -\frac{x_c}{T|_L}\left(5(T - T^*)|_L + 10(T - T^*)|_0\right) \\ &= -\frac{5(T - T^*)|_L + 10(T - T^*)|_0}{\mathcal{H}_0 T|_L} \int_0^L u(z,t) \mathrm{d}z \end{aligned} \tag{85}$$

Heat equation

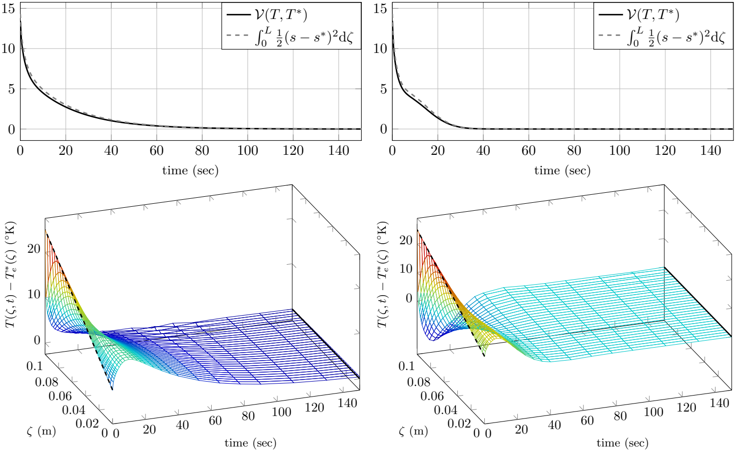


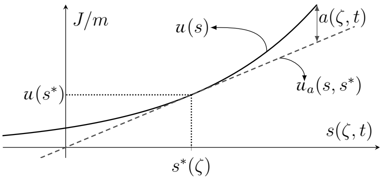
Figure: left: $\phi_L = 5x_c$, $\phi_0 = 0$, right: $\phi_L = 5x_c$, $\phi_0 = 10x_c$

Alternative approach

Idea

- Use the Thermodynamic availability function as closed loop Lyapunov function.

$$\mathcal{A} = \int_0^L (u(s) - u_a(s))\, dz$$



- Use Entropy Assignment to guarantee the convergence of trajectories.

It has been successfully applied to the control of the heat equation. More complex systems (reaction-convection-diffusion systems) are under investigation.

Outline

Conclusions and future works

Conclusion

- We provided an overview on some key results on control of distributed port Hamiltonian systems in the 1D case.
- We detailed a constructive control design technique : energy shaping for boundary/in domain controlled DPS.
- We proposed first ideas on observer design.
- We presented some possibles extensions to irreversible thermodynamic systems.

Conclusions and future works

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- We provided an overview on some key results on control of distributed port Hamiltonian systems in the 1D case.
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- We presented some possibles extensions to irreversible thermodynamic systems.

Future works

- Study of the impact of the distribution of the patches on the achievable performances.
- Control design for a class of non linear PDE systems.
- Extension to 2D DPS.
- Control design for irreversible PHS.

Thank you for your attention !

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