From reversible to irreversible thermodynamic formulations : modelling

Yann Le Gorrec[1] *and Hector Ramirez*

[1]SupMicroTech Besançon FEMTO-ST
Joint work with Bernhard Maschke and Luis Mora.

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Outline

Context, motivation

- In many cutting-edge engineering applications, for example within the field of soft or micro-nano robotics, process control, material sciences, energy production etc ... temperature plays a central role and needs to be explicitly taken into account.

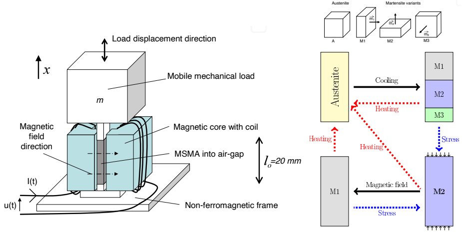

Figure: MSMA actuator.

Cf. Hector's talk on IPHS for finite dimensional systems.

Context, motivation

- Some examples of Distributed Parameter Systems for which the thermal domain plays a central role

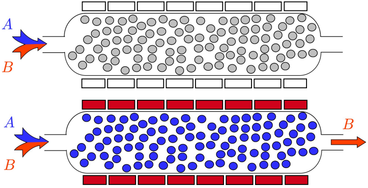

Figure: Adsorption process

Dispersion (column), diffusion (pellet) and non-linear adsorption (crystal)

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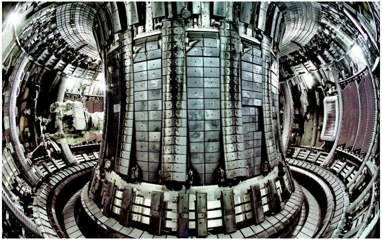

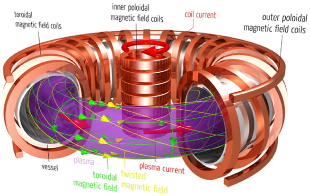

- Several attempts have been made to extend port Hamiltonian and Lagrangian formulations to Irreversible Thermodynamic systems.

Context, motivation

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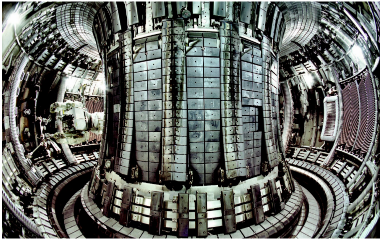

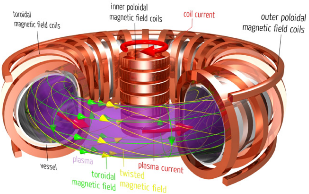

- Several attempts have been made to extend port Hamiltonian and Lagrangian formulations to Irreversible Thermodynamic systems.

In this talk ...

We present some results on the extension of PHS and IPHS formulations to infinite dimensional systems.

Context, motivation

We focus on systems defined on a one dimensional spatial domain.

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We focus on systems defined on a one dimensional spatial domain.

The aim is to generalize PHS formulations

$$\frac{\partial x}{\partial t} = \left(P_0 - G_0 + P_1 \frac{\partial}{\partial \zeta} \right) \mathcal{H} x(\zeta, t)$$

with

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} \tag{1}$$

to irreversible thermodynamic systems ...

????

Outline

The isentropic fluid: the reversible case

We consider a 1-D isentropic fluid in Lagrangian coordinates, also known as *p-system*, with $[a, b] \ni \zeta$, a, $b \in \mathbb{R}$, $a < b$. We choose as state variables

- the specific volume $\phi(\zeta, t)$,
- the velocity $\upsilon(\zeta, t)$ of the fluid.

System of two conservation laws :

$$\frac{\partial \phi}{\partial t}(\zeta, t) = \frac{\partial \upsilon}{\partial \zeta}(\zeta, t)$$
$$\frac{\partial \upsilon}{\partial t}(\zeta, t) = -\frac{\partial p}{\partial \zeta}(\zeta, t)$$

where $p(\phi)$ is the pressure of the fluid. The total energy of the system is given by the sum of the kinetic energy and internal energy:

$$H(\upsilon, \phi) = \int_a^b \left(\frac{1}{2}\upsilon^2 + u(\phi) \right) dz$$

The isentropic fluid: the reversible case

The variational derivative of the total energy yields $\frac{\delta H}{\delta \upsilon} = \upsilon$ and $\frac{\delta H}{\delta \phi} = \frac{\partial u}{\partial \phi} = -p$ and the system may be written as the *Hamiltonian system*

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \end{bmatrix} = P_1 \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix} \right), \qquad \text{with} \quad P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

Considering as input/output (W_B and W_C can be derived from P_1 [Le Gorrec et al., 2005]) :

$$\begin{bmatrix} v \\ y \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \phi}(b) \\ \frac{\delta H}{\delta v}(b) \\ \frac{\delta H}{\delta \phi}(a) \\ \frac{\delta H}{\delta v}(a) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -p(t,b) \\ \upsilon(t,b) \\ -p(t,a) \\ \upsilon(t,a) \end{bmatrix} = \begin{bmatrix} -p(t,b) \\ p(t,a) \\ \upsilon(t,b) \\ \upsilon(t,a) \end{bmatrix}$$

We have

$$\dot{H}(t) = y^{\top}(t) v(t)$$

The non-isentropic fluid: the irreversible case

We now consider some viscous damping τ. The momentum balance reads

$$\frac{\partial \upsilon}{\partial t}(\zeta, t) = -\frac{\partial p}{\partial \zeta}(\zeta, t) - \frac{\partial \tau}{\partial \zeta}(\zeta, t) \tag{3}$$

where

$$\tau = -\hat{\mu}\frac{\partial \upsilon}{\partial \zeta}$$

with $\hat{\mu}$ the viscous damping coefficient. It can be written as a dissipative port Hamiltonian system

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\partial}{\partial \zeta}\left(\hat{\mu}\frac{\partial .}{\partial \zeta}\right) \end{bmatrix}\left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix}\right), \tag{4}$$

The non-isentropic fluid: the irreversible case

Splitting the dissipative operator we have

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix} \right) + \begin{bmatrix} 0 \\ \frac{\partial}{\partial \zeta} \end{bmatrix} \hat{\mu} \begin{bmatrix} 0 & \frac{\partial}{\partial \zeta} \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \end{bmatrix} \right), \tag{5}$$

Which is equivalent to the DAE system:

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ f_e \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ e_e \end{bmatrix} \right), \tag{6}$$

with

$$e_e = \hat{\mu} f_e, \text{ with } \hat{\mu} > 0$$

The non-isentropic fluid: the irreversible case

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The existence of solutions can be proven based on the existence of solutions of the dissipation-free system by direct application of [Le Gorrec et al., 2005]

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with

$$e_e = \hat{\mu} f_e, \text{ with } \hat{\mu} > 0$$

The existence of solutions can be proven based on the existence of solutions of the dissipation-free system by direct application of [Le Gorrec et al., 2005] but not stability.

The non-isentropic fluid: the irreversible case

We can account for the thermal domain by considering Gibbs' equation

$$du = -pd\phi + Tds$$

where s denotes the entropy density and T the temperature. The total energy of the system is still the sum of the kinetic and the internal energy but now depends on s

$$H(\upsilon,\,\phi,\,s) = \int_a^b \left(\frac{1}{2}\upsilon^2 + u(\phi, s)\right) dz$$

From the conservation of the total energy and Gibbs' equation $\frac{\partial u}{\partial s} = T$ we get

$$\frac{\partial s}{\partial t}(\zeta, t) = \frac{\hat{\mu}}{T}\left(\frac{\partial \upsilon}{\partial \zeta}\right)^2 (\zeta, t)$$

The non-isentropic fluid: the irreversible case

The system of balance equations may be written as the quasi-Hamiltonian system

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ \frac{\partial s}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial (\cdot)}{\partial \zeta} & 0 \\ \frac{\partial (\cdot)}{\partial \zeta} & 0 & \frac{\partial}{\partial \zeta}\left(\frac{\hat{\mu}}{T} \left(\frac{\partial \upsilon}{\partial \zeta} \right) (\cdot) \right) \\ 0 & \frac{\hat{\mu}}{T} \left(\frac{\partial \upsilon}{\partial \zeta} \right) \frac{\partial (\cdot)}{\partial \zeta} & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ \frac{\delta H}{\delta s} \end{bmatrix} \right)$$

Question: Is this operator formally skew symmetric ?

The non-isentropic fluid: the irreversible case

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Question: Can you write down the energy balance ? What are the possible boundary port variables ?

The non-isentropic fluid: the irreversible case

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Question: Is this operator formally skew symmetric ?
Question: Can you write down the energy balance ? What are the possible boundary port variables ?
Question: Can you write down the entropy balance ? What are the possible boundary port variables ?

The non-isentropic fluid: the irreversible case

The system of balance equations may be written as the quasi-Hamiltonian system

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Answer: Yes ! even if the differential operator is modulated.

The non-isentropic fluid: the irreversible case

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$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ \frac{\partial s}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial (\cdot)}{\partial \zeta} & 0 \\ \frac{\partial (\cdot)}{\partial \zeta} & 0 & \frac{\partial}{\partial \zeta}\left(\frac{\hat{\mu}}{T} \left(\frac{\partial \upsilon}{\partial \zeta} \right) (\cdot) \right) \\ 0 & \frac{\hat{\mu}}{T} \left(\frac{\partial \upsilon}{\partial \zeta} \right) \frac{\partial (\cdot)}{\partial \zeta} & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ \frac{\delta H}{\delta s} \end{bmatrix} \right)$$

Answer: Yes ! even if the differential operator is modulated.
Answer: In case we do not have homogeneous BC:

$$\frac{dH}{dt} = y^T \nu$$

and

$$\frac{dS}{dt} = \int_a^b \sigma d\zeta \geq 0$$

Outline

IPHS : General formulation

We introduce the Boundary Controlled Irreversible Port Hamiltonian System (BC-IPHS) defined on a 1D spatial domain $\zeta \in [a, b]$, a, $b \in \mathbb{R}$, $a < b$. The state variables of the system are the $n+1$ *extensive variables*. The following partition of the state vector $\mathbf{x} \in \mathbb{R}^{n+1}$ shall be considered: the first n variables by $x = [q_1, \ldots, q_n]^\top \in \mathbb{R}^n$ and the entropy density by $s \in \mathbb{R}$. Gibbs' equation is equivalent to the existence of an energy functional

$$H(x, s) = \int_a^b h\left(x(\zeta), s(\zeta)\right) d\zeta \tag{7}$$

where $h(x, s)$ is the energy density function. The total entropy functional is denoted by

$$S(t) = \int_a^b s(\zeta, t) d\zeta \tag{8}$$

IPHS : General formulation

An infinite dimensional IPHS undergoing m irreversible processes is defined by

$$\frac{\partial}{\partial t}\begin{bmatrix} x(\zeta,t) \\ s(\zeta,t)\end{bmatrix} = \begin{bmatrix} P_0 & G_0 \mathbf{R_0} \\ -\mathbf{R_0}^\top G_0^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t)\end{bmatrix} + \begin{bmatrix} P_1 \frac{\partial(.)}{\partial \zeta} & \frac{\partial(G_1 \mathbf{R_1} .)}{\partial \zeta} \\ \mathbf{R_1}^\top G_1^\top \frac{\partial(.)}{\partial \zeta} & g_s \mathbf{r_s} \frac{\partial(.)}{\partial \zeta} + \frac{\partial(g_s \mathbf{r_s} .)}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t)\end{bmatrix} \tag{9}$$

where $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$, $G_0 \in \mathbb{R}^{n\times m}$, $G_1 \in \mathbb{R}^{n\times m}$ with $m \leq n$ with $\mathbf{R_l}\left(\mathbf{x}, \frac{\delta H}{\delta \mathbf{x}}\right) \in \mathbb{R}^{m\times 1}$, $l = 0, 1$, defined by

$$R_{0,i} = \gamma_{0,i}\left(x, z, \frac{\delta H}{\delta x}\right)\{S|G_0(:,i)|H\}$$

$$R_{1,i} = \gamma_{1,i}\left(x, z, \frac{\delta H}{\delta x}\right)\left\{S|G_1(:,i)\frac{\partial}{\partial \zeta}|H\right\}$$

and

$$r_s = \gamma_s\left(x, z, \frac{\delta H}{\delta x}\right)\{S|H\}$$

and $\gamma_{k,i}\left(x, z, \frac{\delta H}{\delta x}\right) > 0,\ k = 0, 1;\ i \in \{1, \ldots m\}$, $\gamma_s\left(x, z, \frac{\delta H}{\delta x}\right) > 0$ and $g_s(x)$,

IPHS : General formulation

For any two functionals H_1 and H_2 of the type (7) and for any matrix differential operator $\mathcal{G}$ we define the pseudo-brackets

$$\begin{aligned} \{H_1|\mathcal{G}|H_2\} &= \begin{bmatrix} \frac{\delta H_1}{\delta x} \\ \frac{\delta H_1}{\delta s} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H_2}{\delta x} \\ \frac{\delta H_2}{\delta s} \end{bmatrix}, \\ \{H_1|H_2\} &= \frac{\delta H_1}{\delta s}^{\top} \left(\frac{\partial}{\partial \zeta} \frac{\delta H_2}{\delta s} \right) \end{aligned} \quad (10)$$

where $\mathcal{G}^*$ denotes the formal adjoint operator of $\mathcal{G}$.

IPHS : General formulation

Remark 1:

Setting the matrices P_1 and G_1 to zero, reduces the PDE (11) to

$$\frac{d}{dt}\begin{bmatrix} x(\zeta,t) \\ s(\zeta,t) \end{bmatrix} = \begin{bmatrix} P_0 & G_0\mathbf{R_0}(\mathbf{x}) \\ -\mathbf{R_0}(\mathbf{x})^\top G_0^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t) \end{bmatrix}$$

which is formally the definition of finite-dimensional IPHS in [Ramirez et al., 2013a, Ramirez et al., 2013b] for the case $m = 1$ or [Ramirez et al., 2014, Ramirez et al., 2016] for $m > 1$. In this sense the previous definition is an infinite-dimensional extension of the definition of IPHS.

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Remark 2:

Setting the matrices G_0 and G_1 to zero reduces the PDE (11) to

$$\frac{d}{dt}\begin{bmatrix} x(\zeta,t) \\ s(\zeta,t) \end{bmatrix} = \begin{bmatrix} P_0 + P_1 \frac{\partial(.)}{\partial \zeta} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t) \end{bmatrix}$$

which is formally the definition of infinite-dimensional PHS.

IPHS : General formulation

Definition 1

A Boundary Controlled IPHS (BC-IPHS) is an infinite dimensional IPHS

$$\frac{\partial}{\partial t}\begin{bmatrix} x(\zeta,t) \\ s(\zeta,t)\end{bmatrix} = \begin{bmatrix} P_0 & G_0\mathbf{R_0} \\ -\mathbf{R_0}^\top G_0^\top & 0 \end{bmatrix}\begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t)\end{bmatrix} + \begin{bmatrix} P_1\frac{\partial(.)}{\partial\zeta} & \frac{\partial(G_1\mathbf{R_1}.)}{\partial\zeta} \\ \mathbf{R_1}^\top G_1^\top\frac{\partial(.)}{\partial\zeta} & g_s\mathbf{r_s}\frac{\partial(.)}{\partial\zeta} + \frac{\partial(g_s\mathbf{r_s}.)}{\partial\zeta}\end{bmatrix}\begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \frac{\delta H}{\delta s}(\zeta,t)\end{bmatrix} \quad (11)$$

Augmented with the boundary port variables

$$v(t) = W_B\begin{bmatrix} e(t,b) \\ e(t,a)\end{bmatrix}, \qquad y(t) = W_C\begin{bmatrix} e(t,b) \\ e(t,a)\end{bmatrix} \quad (12)$$

as linear functions of the modified effort variable

$$e(\zeta,t) = \begin{bmatrix} \frac{\delta H}{\delta x}(\zeta,t) \\ \mathbf{R}(\mathbf{x},\frac{\delta H}{\delta \mathbf{x}})\,\frac{\delta H}{\delta s}(\zeta,t)\end{bmatrix}, \text{ with } \mathbf{R}\left(\mathbf{x},\frac{\delta H}{\delta \mathbf{x}}\right) = \begin{bmatrix} 1 & \mathbf{R_1}(\mathbf{x},\frac{\delta \mathbf{H}}{\delta \mathbf{x}}) & \mathbf{r_s}(\mathbf{x},\frac{\delta \mathbf{H}}{\delta \mathbf{x}})\end{bmatrix}^\top \quad (13)$$

IPHS : General formulation

Furthermore

$$W_B = \left[\frac{1}{\sqrt{2}} \left(\Xi_2 + \Xi_1 P_{ep}\right) M_p \quad \frac{1}{\sqrt{2}} \left(\Xi_2 - \Xi_1 P_{ep}\right) M_p \right],$$
$$W_C = \left[\frac{1}{\sqrt{2}} \left(\Xi_1 + \Xi_2 P_{ep}\right) M_p \quad \frac{1}{\sqrt{2}} \left(\Xi_1 - \Xi_2 P_{ep}\right) M_p \right],$$

where $M_p = \left(M^\top M\right)^{-1} M^\top$, $P_{ep} = M^\top P_e M$ and $M \in \mathbb{R}^{(n+m+2)\times k}$ is spanning the columns of $P_e \in \mathbb{R}^{n+m+2}$ of rank k, defined by

$$P_e = \begin{bmatrix} P_1 & 0 & G_1 & 0 \\ 0 & 0 & 0 & g_s \\ G_1^\top & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 \end{bmatrix} \tag{14}$$

and where Ξ_1 and Ξ_2 in $\mathbb{R}^{k\times k}$ satisfy $\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0$ and $\Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I$.

IPHS : General formulation

First law of Thermodynamics

The total energy balance is

$$\dot{H} = y(t)^\top v(t)$$

which leads, when the input is set to zero, to $\dot{H} = 0$ in accordance with the first law of Thermodynamics.

Sketch of the proof

$$\frac{dH}{dt} = \int_a^b \begin{bmatrix} \frac{\delta H}{\delta x}^T & \frac{\delta H}{\delta s} \end{bmatrix} \begin{bmatrix} \frac{dx}{\delta t} \\ \frac{ds}{\delta t} \end{bmatrix} d\zeta = \int_a^b \begin{bmatrix} \frac{\delta H}{\delta x}^T & \frac{\delta H}{\delta s} \end{bmatrix} \mathcal{J}_e \begin{bmatrix} \frac{\delta H}{\delta x} \\ \frac{\delta H}{\delta s} \end{bmatrix} d\zeta = v^T y$$

IPHS : General formulation

Second law of Thermodynamics

The total entropy balance is given by

$$\dot{S} = \int_a^b \sigma_t d\zeta + y_S^\top v_s$$

where y_s and v_s are the entropy conjugated input/output and σ_t is the total internal entropy production. This leads, when the input is set to zero, to $\dot{S} = \int_a^b \sigma_t d\zeta \geq 0$ in accordance with the second law of Thermodynamics.

Sketch of the proof

$$\begin{aligned}\dot{S} &= \int_a^b \frac{\partial s}{\partial t} d\zeta \\ &= \int_a^b \left(\mathbf{R_0}(\mathbf{x})^\top G_0^\top \frac{\delta H}{\delta x} + \mathbf{R_1}(\mathbf{x})^\top G_1^\top \frac{\partial}{\partial \zeta} \frac{\delta H}{\delta x} + \right. \\ & \left. g_s \mathbf{r_s}(\mathbf{x}) \frac{\partial}{\partial \zeta} \frac{\delta H}{\delta s} + \frac{\partial}{\partial \zeta} \left(g_s \mathbf{r_s}(\mathbf{x}) \frac{\delta H}{\delta x} \right) \right) d\zeta \\ &= \int_a^b \sigma_t d\zeta - (f_s(b,t) - f_s(a,t))\end{aligned}$$

Back to 1D fluid

Recalling the 1D fluid model

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ \frac{\partial s}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial (\cdot)}{\partial \zeta} & 0 \\ \frac{\partial (\cdot)}{\partial \zeta} & 0 & \frac{\partial}{\partial \zeta}\left(\frac{\hat{\mu}}{T}\left(\frac{\partial \upsilon}{\partial \zeta}\right)(\cdot)\right) \\ 0 & \frac{\hat{\mu}}{T}\left(\frac{\partial \upsilon}{\partial \zeta}\right)\frac{\partial (\cdot)}{\partial \zeta} & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \phi} \\ \frac{\delta H}{\delta \upsilon} \\ \frac{\delta H}{\delta s} \end{bmatrix} \right)$$

$P_0 = 0, G_0 = 0, g_s = 0, P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $x = \begin{bmatrix} \phi \\ \upsilon \end{bmatrix}$ and $R_{11} = \gamma_1 \{S | G_1(:,1) \frac{\partial}{\partial z} | H\}$ with $\gamma_1 = \frac{\hat{\mu}}{T} > 0$. In this case $n = 2$, $m = 1$

Back to 1D fluid

The boundary port variables may be computed as follows, starting with

$$P_e = \begin{bmatrix} P_1 & 0 & G_1 & 0 \\ 0 & 0 & 0 & g_s \\ G_1^\top & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

of rank $k = 2$ which gives $M = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top$, $M_P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $P_{ep} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Choosing the parametrization

$$\Xi_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Xi_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

define the following boundary inputs and outputs

$$v(t) = \begin{bmatrix} -p(t,b) + \frac{\hat{\mu}}{T} \frac{\partial \upsilon}{\partial z}(t,b) \\ p(t,a) - \frac{\hat{\mu}}{T} \frac{\partial \upsilon}{\partial z}(t,a) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} \upsilon(t,b) \\ \upsilon(t,a) \end{bmatrix}.$$

1D fluid in Eulerian coordinates

If we now consider a 1-D isentropic fluid in Eulerian coordinates, with $[a, b] \ni \zeta$, a, $b \in \mathbb{R}$, $a < b$. We choose as state variables

- the mass density $\rho(\zeta, t)$,
- the velocity $\upsilon(\zeta, t)$ of the fluid.

System of two conservation laws (coming from the use of material derivative $\frac{D}{Dt}(.) = \frac{\partial}{\partial t}(.) + \upsilon \frac{\partial}{\partial \zeta}(.)$) and from Gibbs equation:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}(\rho \upsilon)(\zeta, t) \\
\frac{\partial \upsilon}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta}\left(\frac{1}{2}\upsilon^2 + u + \frac{p}{\rho}\right)(\zeta, t) + \frac{T}{\rho}\frac{\partial s}{\partial \zeta}(\zeta, t) - \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\frac{\tau}{\rho}\right)(\zeta, t)
\end{aligned}
$$

and

$$
\frac{\partial s}{\partial t}(\zeta, t) = -\upsilon \frac{\partial s}{\partial \zeta}(\zeta, t) - \frac{\tau}{\rho T}\frac{\partial \upsilon}{\partial \zeta}(\zeta, t)
$$

1D fluid in Eulerian coordinates

Leading to the IPHS representation

$$\begin{bmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial \upsilon}{\partial t} \\ \frac{\partial s}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial (\cdot)}{\partial \zeta} & 0 \\ -\frac{\partial (\cdot)}{\partial \zeta} & 0 & \frac{1}{\rho}\frac{\partial s}{\partial \zeta}(\zeta,t) + \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\frac{\mu}{\rho T}\frac{\partial \upsilon}{\partial \zeta}\,\cdot\right) \\ 0 & -\frac{1}{\rho}\frac{\partial s}{\partial \zeta}(\zeta,t) + \frac{\mu}{\rho T}\frac{\partial \upsilon}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{1}{\rho}\,\cdot\right) & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{\delta H}{\delta \rho} \\ \frac{\delta H}{\delta \upsilon} \\ \frac{\delta H}{\delta s} \end{bmatrix} \right)$$

with

$$f_\partial = \begin{bmatrix} -\upsilon(b,t) \\ \upsilon(a,t) \end{bmatrix} \text{ and } e_\partial = \begin{bmatrix} \left(\rho\left(\frac{1}{2}\upsilon^2 + h\right)\right)(b,t) - \mu\frac{\partial \upsilon}{\partial \zeta}(b,t) \\ \left(\rho\left(\frac{1}{2}\upsilon^2 + h\right)\right)(b,t) - \mu\frac{\partial \upsilon}{\partial \zeta}(a,t) \end{bmatrix}$$

and $h = u + \frac{p}{\rho}$

Heat equation

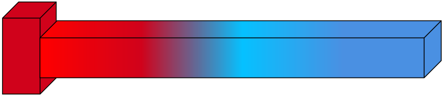


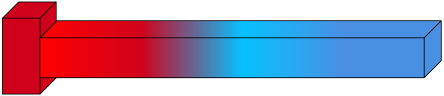
Figure: Heat conduction in a bar

Balance equation on u

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial \zeta}\left(-\lambda \frac{\partial T}{\partial \zeta}\right)$$

where λ denotes the heat conduction coefficient.
Question : Deduce from Gibbs' equation $du = Tds$ the IPHS formulation of the heat equation.

Heat equation



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Question : Deduce from Gibbs' equation $du = Tds$ the IPHS formulation of the heat equation.
Question : Write the balance equation on the energy/entropy.

Heat equation

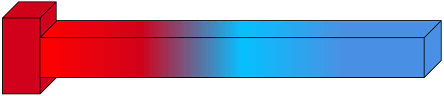

Figure: Heat conduction in a bar

Balance equation on u

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial \zeta}\left(-\lambda \frac{\partial T}{\partial \zeta}\right)$$

where λ denotes the heat conduction coefficient. From Gibbs' equation $du = Tds$ and

$$\frac{\partial s}{\partial t} = -\frac{1}{T}\frac{\partial}{\partial \zeta}\left(-\lambda \frac{\partial T}{\partial \zeta}\right)$$

or alternatively

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial \zeta}\left(\frac{\lambda}{T}\frac{\partial T}{\partial \zeta}\right) + \frac{\lambda}{T^2}\left(\frac{\partial T}{\partial \zeta}\right)^2$$

One can notice that : $T = \frac{\delta U}{\delta s}$ where $U = \int_a^b u d\zeta$.

Heat equation

IPHS formulation

$$\frac{\partial s}{\partial t} = \frac{\lambda}{T^2}\frac{\partial T}{\partial \zeta}\frac{\partial}{\partial \zeta}\left(\frac{\delta U}{\delta s}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\lambda}{T^2}\frac{\partial T}{\partial \zeta}\left(\frac{\delta U}{\delta s}\right)\right)$$

which is equivalent to (11) where $P_0 = 0$, $P_1 = 0$, $G_0 = 0$, $G_1 = 0$, $g_s = 1$ and $r_s = \gamma_s\{S|U\}$ with $\gamma_s = \frac{\lambda}{T^2}$ and $\{S|U\} = \frac{\partial T}{\partial \zeta}$. In this case $P_e = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $n = 1$ and $m = 1$. Choosing $\Xi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Xi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ the boundary inputs and outputs of the system are

$$v(t) = \begin{bmatrix} \left(\frac{\lambda_s}{T}\frac{\partial T}{\partial \zeta}\right)(t,b) \\ -\left(\frac{\lambda_s}{T}\frac{\partial T}{\partial \zeta}\right)(t,a) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} T(t,b) \\ T(t,a) \end{bmatrix},$$

respectively the entropy flux and the temperature at each boundary.

Outline

Context, motivation

A simple but instructive example

Infinite dimensional irreversible port Hamiltonian systems (IPHS)

Conclusions

In this talk we have:

- introduced a new class of boundary controlled IPHS.
- illustrated it on some examples (fluid and heat equations).

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- introduced a new class of boundary controlled IPHS.
- illustrated it on some examples (fluid and heat equations).

Current research lines:

- extend IPHS formulation to multidimensional systems such as fluids.
- extend the traditional control design techniques to thermally controlled BC-IPHS.
- use this formalism to overcome some traditional difficulties associated to irreversible Thermodynamics.

Thank you for your attention !

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