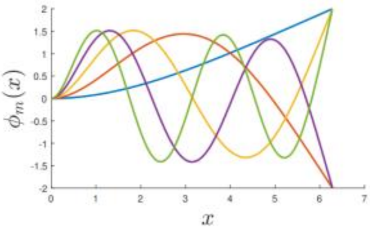
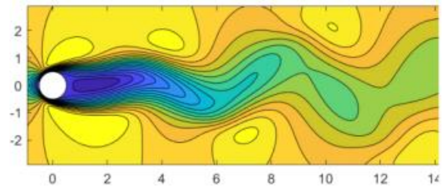
Plan for Lectures 15th & 16th November

1. **Modelling of Fluids and Structures**

   Structures: PDE examples of strings, Euler-Bernoulli beams.
   a. The concept of a vibration mode shape
   b. Reduced order models



   Fluids: Navier-Stokes equations
   a. Coherent structures for fluids
   b. Data-driven identification of coherent structures
   c. Data-driven dynamics models



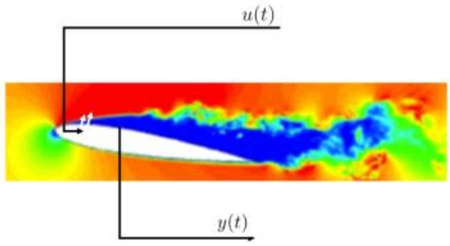
2. **Stability of nonlinear systems**

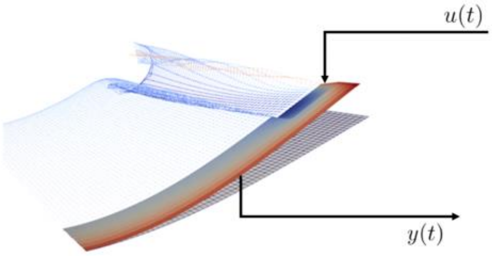
   a. *Lyapunov stability theory*
   b. Discrete and continuous time systems

3. **Model Predictive Control**

   A brief introduction to MPC and its closed-loop stability.

Modelling and Control of Flexible Structures interacting with Fluids





Modelling:

$$\dot{x}(t) = f(x(t), u(t)) \qquad x_{k+1} = f(x_k, u_k)$$
$$y(t) = h(x(t)) \qquad y_k = h(x_k)$$

Question: How can we create usable models from PDEs?

Stability:

$$\lim_{t \to \infty} x(t) = 0$$

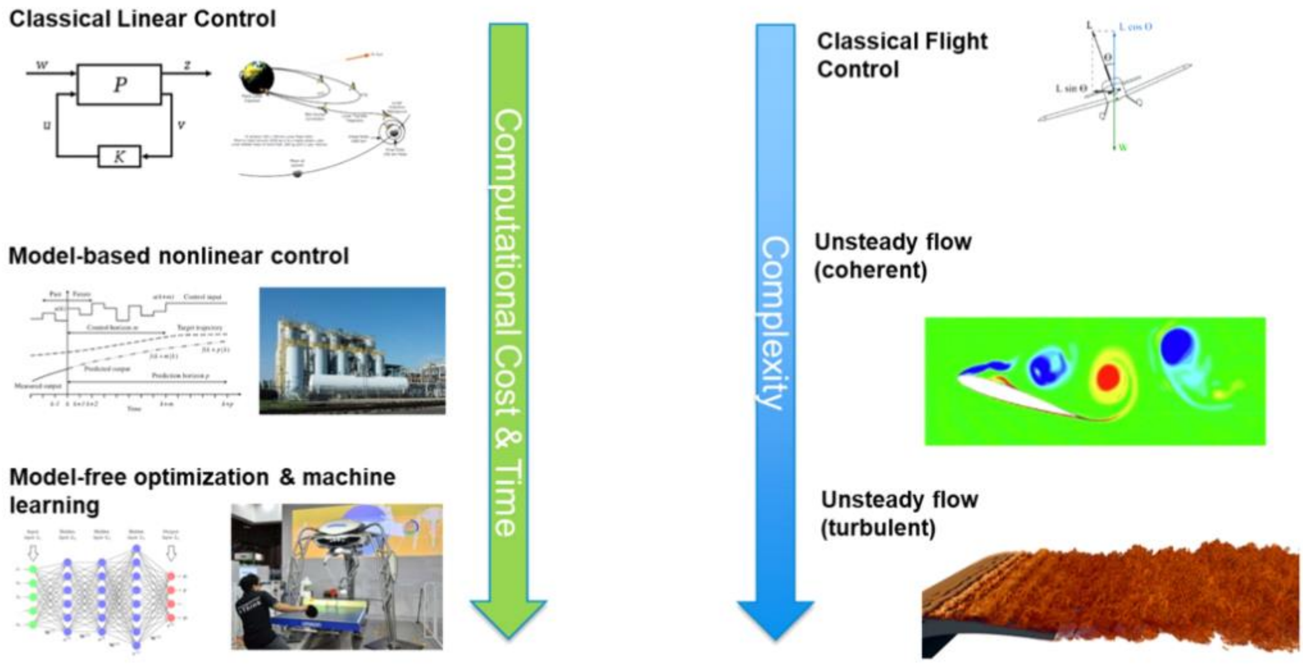
Question: How can we characterise or check stability?

Control:

$$u(t) = f(y(t))$$

Question: Can stability be enhanced by (feedback) control

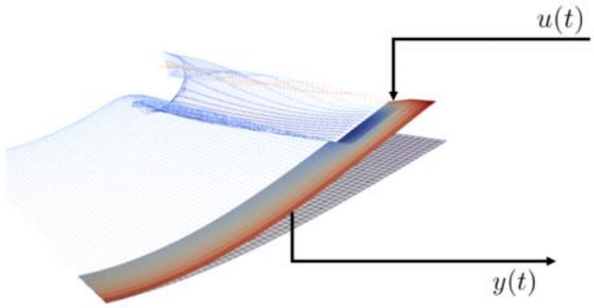
Some challenges of controlling fluid-structure interactions



1.1 Continuous time models

- We will seek to construct finite-dimensional state-space models in these lectures:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t))$$



where

$x(t) \in \mathbb{R}^n$ is the *state* of the system

$u(t) \in \mathbb{R}^u$ is the control input

$y(t) \in \mathbb{R}^p$ are the measured outputs

and

$$f : \mathbb{R}^n \times \mathbb{R}^u \to \mathbb{R}^n$$

$$h : \mathbb{R}^n \to \mathbb{R}^p$$

1.1 Linear Control Systems

$$A \in \mathbb{R}^{n\times n}$$
$$B \in \mathbb{R}^{n\times u}$$
$$C \in \mathbb{R}^{p\times n}$$

- Suppose

$$f(x,u) = Ax + Bu, \qquad h(x) = Cx$$

then

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Solution

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}Bu(s)ds$$

with

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots$$

1.1 Discrete Time Control Systems

Suppose that

$$x_{k+1} = f(x_k, u_k), \qquad y_k = h(x_k)$$

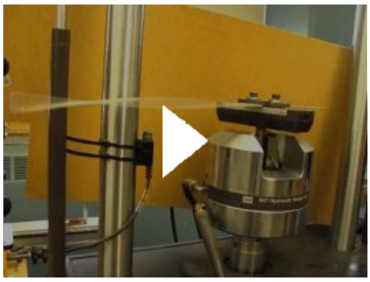
Interpretation: x_k, u_k, y_k are the states, inputs and measurements at times

$$t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$$

Discrete time systems are important for numerical implementation and are naturally when identifying models from data

1.2 Two examples from Structural Mechanics

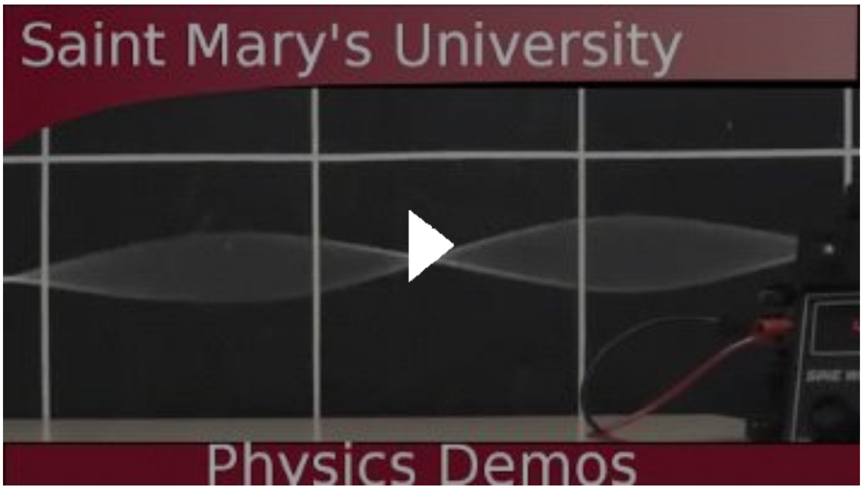
We now look at two PDE models from structural mechanics:



a. A model for a thin elastic string

b. The Euler-Bernoulli model for beam bending

In looking at these examples we will introduce the concept of a **vibration mode** which will be used as a natural basis for creating reduced order models



1.2 A vibrating string

Example 1 (Vibrating String). *Let $u(x,t)$ be the vertical displacement of a string at position $0 \le x \le L$ and at time $t \ge 0$. The string is held fixed at its endpoints $x = 0, x = L$ and is assumed to satisfy the PDE*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \\ u(0,t) &= 0, \qquad t \ge 0 \\ u(L,t) &= 0, \qquad t \ge 0. \end{aligned}$$

Suppose that the string is initially at rest and that the initial displacement and velocity of the string are given by

$$u(x,0) = h(x), \quad \frac{\partial u}{\partial t}(x,0) = v(x), \qquad 0 \le x \le L.$$

Find the displacement $u(x,t)$ for all $t \ge 0$.

1.2 Vibration Modes & Reduced Order Models

- Have decomposed solution into **vibration modes:**

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t + P_n) \sin(\omega_n x)$$

- This implies a natural way to create reduced order models by using an finite dimensional series expansion

$$u(x,t) \approx \sum_{i=1}^{n} q_{1i}(t)\phi_i(x)$$

1.2 Reduced order models

- To find a reduced order model assume a finite dimensional series and substitute into the PDE:

$$u(x,t) \approx \sum_{i=1}^{n} q_{1i}(t)\phi_i(x)$$

→ sub

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0, \qquad t \geq 0$$

$$u(L,t) = 0, \qquad t \geq 0.$$

1.2 An alternative view

- Substitution into the PDE works but depends on **already knowing** a good choice for the vibration mode shapes

$$\phi_n(x) := \sin(\omega_n x)$$

- An alternative is to re-write the PDE in first order form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix}$$

The mode shapes then arise naturally as **eigenfunctions** of the operator generating this linear PDE

Example 3. *Consider the operator*

$$A = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

Let $\omega_n = \frac{\pi n}{L}$. *Show that* $\lambda_n = i\omega_n$ *are eigenvalues of* A *with eigenfunctions given by*

$$\Phi_n(x) = \begin{pmatrix} \phi_n(x) \\ \psi_n(x) \end{pmatrix} = \begin{pmatrix} \sin(\omega_n x) \\ i\omega_n \sin(\omega_n x) \end{pmatrix}.$$

1.2 The string equation

Example 4. Question: *In the construction of the "reduced order" models described above, any subset of the vibration modes* $\phi_n(x) = \sin(\omega_n x)$ *could have been chosen to create a reduced-order model of a given size. Why might it be a good idea to select that "first n" modes?*

Example 5. Question: *Suppose that the string is initially at rest and that its initial displacement has a parabolic distribution*

$$u(x,0) = x(L-x), \qquad 0 \le x \le L.$$

What is the amplitude of the states $q_{1i}(t), q_{2i}(t)$ *of the full-order series solution to the PDE?*

1.3 The Euler Bernoulli beam

Euler Bernoulli ...

Example 5 (Cantilever Beam). *Let $u(x,t)$ be the vertical displacement of a beam at position $0 \leq x \leq L$ and at time $t \geq 0$. The beam is clamed at the end-point $x = 0$ and can move freely at the other end $x = L$. The displacement is assumed to satisfy the PDE*

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

$$u(0,t) = 0 = u_x(0,t), \qquad t \geq 0$$

$$u_{xx}(L,t) = 0 = u_{xxx}(L,t), \qquad t \geq 0.$$

Find a reduced order model for the beam's dynamics.

1.3 Euler Bernoulli Vibration Modes

$$\phi(x) = \omega^2 \frac{d^4\phi}{dx^4}(x) \xrightarrow[\text{conditions}]{\text{boundary}} \cosh(\sqrt{\omega}L)\cos(\sqrt{\omega}L) + 1 = 0.$$

Question: what can be said about the natural frequencies satisfying this equation?

1.3 Euler Bernoulli Vibration Modes

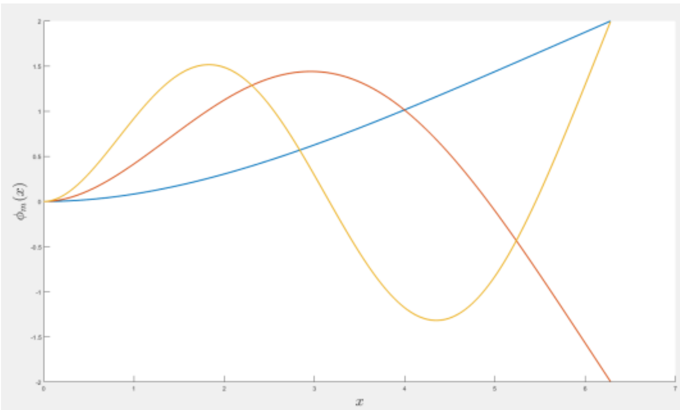
Vibration modes of an Euler-Bernoulli beam:

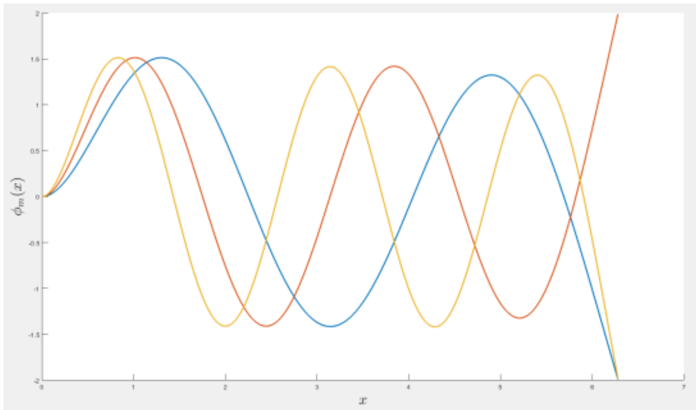
$$\phi_n(x) = \left[\cosh\left(\frac{\beta_n \pi x}{L}\right) - \cos\left(\frac{\beta_n \pi x}{L}\right)\right] - \left[\frac{\cosh(\beta_n \pi) + \cos(\beta_n \pi)}{\sinh(\beta_n \pi) + \sin(\beta_n \pi)}\right]\left[\sinh\left(\frac{\beta_n \pi x}{L}\right) - \sin\left(\frac{\beta_n \pi x}{L}\right)\right]$$

- The natural frequencies $\omega_n = \frac{\pi^2}{L^2}\beta_n^2$ are solutions to

$$\cosh(\sqrt{\omega}L)\cos(\sqrt{\omega}L) + 1 = 0.$$

- The first six mode shapes are plotted below





- A reduced-order mode for the flow can then be created as for the case of the string equation by letting

$$u(x,t) = \sum_{i=1}^{n} q_{1i}(t)\phi_n(x)$$

sub

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

$$u(0,t) = 0 = u_x(0,t), \qquad t \geq 0$$

$$u_{xx}(0,t) = 0 = u_{xxx}(0,t), \qquad t \geq 0.$$

1.3 Mode Orthogonality

Example 8. *Show that the Euler-Bernoulli mode shapes for a clamped beam are orthogonal.*

1.3 Reduced order beam models with control

Example 7. *Consider a cantilever Euler-Bernoulli beam upon which a force $u(t)$ is applied to a section $a-\epsilon \le x \le a+\epsilon$ of the beam. We assume this is modelled by extending the PDE to be*

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} &= g(x)u(t) \\ w(0,t) = 0 &= w_x(0,t), \qquad t \ge 0 \\ w_{xx}(0,t) = 0 &= w_{xxx}(0,t), \qquad t \ge 0. \end{aligned}$$

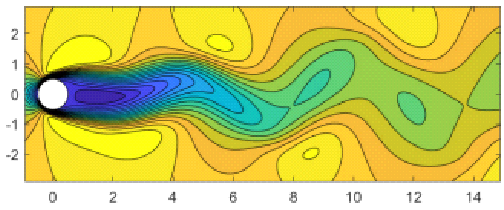
where

$$g(x) = \begin{cases} \frac{1}{2\epsilon} & a-\epsilon \le x \le a+\epsilon, \\ 0 & \textit{otherwise} \end{cases}.$$

Construct a reduced-order model for the controlled system.

1.4 Reduced Order Models For Fluid Flows

- For fluids modelling the challenge is the nonlinear Navier-Stokes equations!



cyl_flow - Shortcut

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re}\Delta \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

- System state is the flow velocity

$$\boldsymbol{u} = \begin{pmatrix} u(\boldsymbol{x},t) \\ v(\boldsymbol{x},t) \\ w(\boldsymbol{x},t) \end{pmatrix}, \qquad \boldsymbol{x} \in \Omega$$

We will not attempt to study this PDE analytically. Instead, we will create reduced-order models for flows **from data.**

To do this, we use the same philosophy as for the string and beam equations and assume

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{i=1}^{n} x_i(t)\Phi_i(\boldsymbol{x}), \qquad t \geq 0, \boldsymbol{x} \in \Omega.$$

Question: what are ``good'' mode shapes for turbulent fluid flow?

1.4.1 Coherent Structures in Fluid Flows

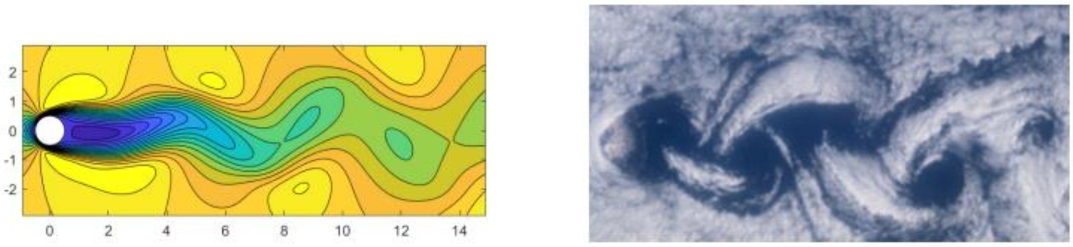
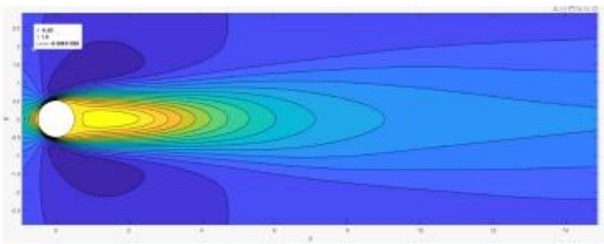


Figure 1: Numerical simulation of 2D flow past a circular cylinder (left) and atmospheric Von-Karman vortex shedding for flow past an island! (right).

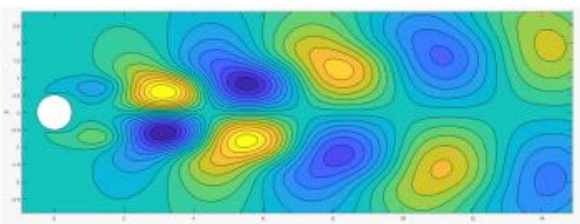
$$\boldsymbol{u}(\boldsymbol{z},t) = \begin{pmatrix} u(\boldsymbol{z},t) \\ v(\boldsymbol{z},t) \end{pmatrix}$$

- The following flow fields which are "good'' choices for coherent structures for this flow

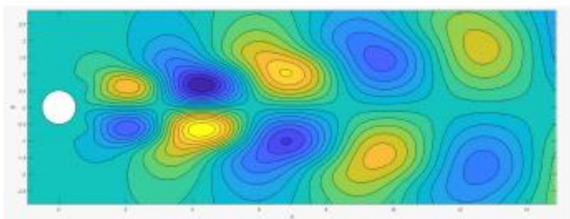
$$\bar{\boldsymbol{u}}(\boldsymbol{z},t) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{u}(\boldsymbol{z},t)dt, \qquad \boldsymbol{z} \in \Omega,$$



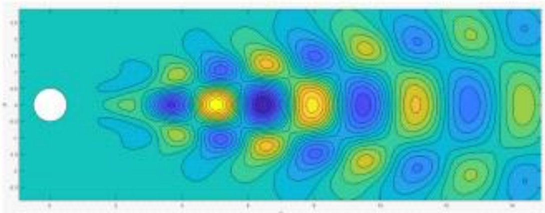
(a) $\bar{u}(\boldsymbol{z})$



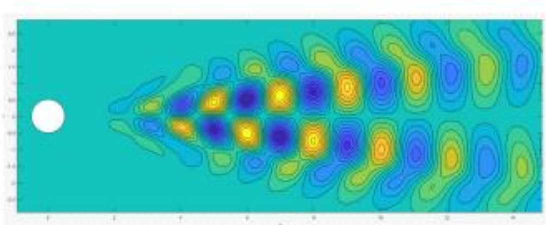
(b) $\Phi_1(\boldsymbol{z})$



(c) $\Phi_2(\boldsymbol{z})$



(d) $\Phi_3(\boldsymbol{z})$

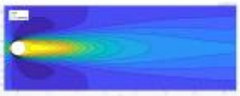


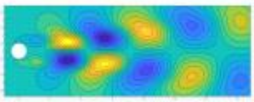
(e) $\Phi_4(\boldsymbol{z})$

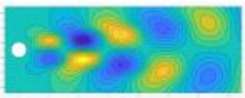
- Will see subsequently how to compute these structures. If we can do this the idea is to approximate the flow using a series expansion

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{z},t) &\approx \bar{\boldsymbol{u}}(\boldsymbol{z}) + \sum_{j=1}^{N} x_j(t)\boldsymbol{\Phi}_j(\boldsymbol{z}) \\ &= \bar{\boldsymbol{u}}(\boldsymbol{z}) + x_1(t)\boldsymbol{\Phi}_1(\boldsymbol{z}) + \cdots + x_N(t)\boldsymbol{\Phi}_N(\boldsymbol{z})\end{aligned}$$

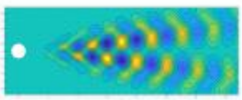
- In pictures:

$\boldsymbol{u}(\boldsymbol{z},t) =$ 

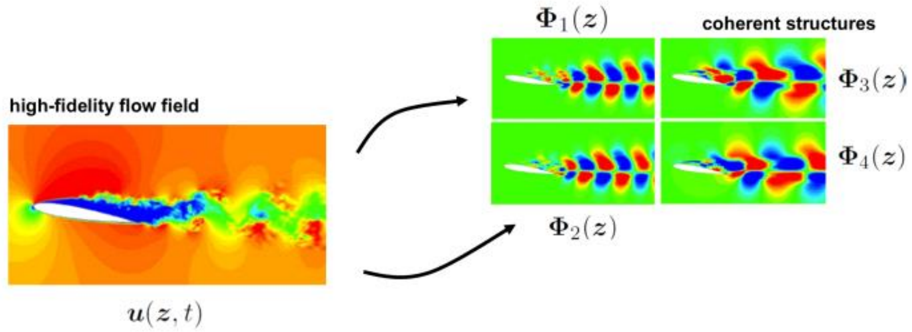
$+ \quad x_1(t)$ 

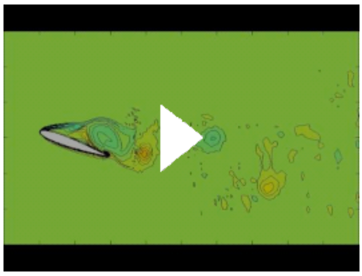
$+ \quad x_2(t)$ 

$\vdots$

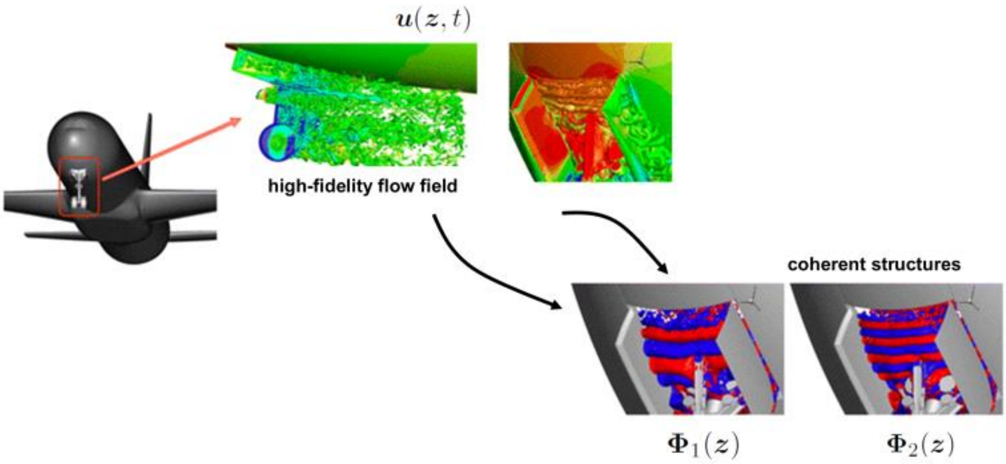
$+ \quad x_N(t)$ 

1.4.1 Coherent Structures In Fluid Flows





(a) 2D aerofoil wake



(b) 3D turbulent flow past a landing gear

1.5 Coherent Structures From Data

- Suppose that have data of a fluid flow velocity

$$\boldsymbol{u}(\boldsymbol{z}_i, t)$$

At fixed spatial locations

$$\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_p.$$

- This is referred to as a **snapshot** of the flow

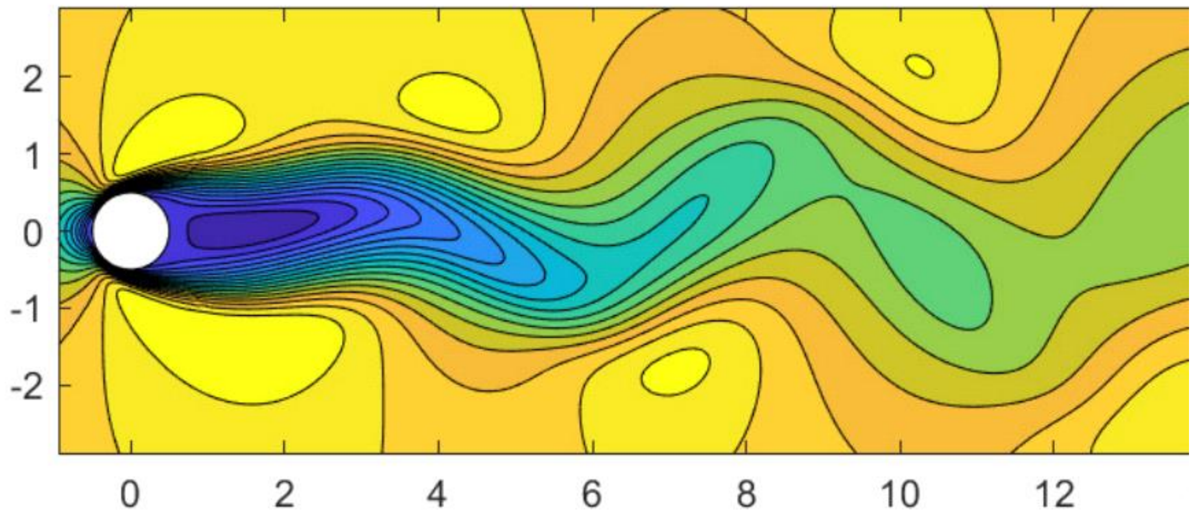


Figure 4: Snapshot of 2D flow past a circular cylinder.

1.5 Coherent Structures From Data

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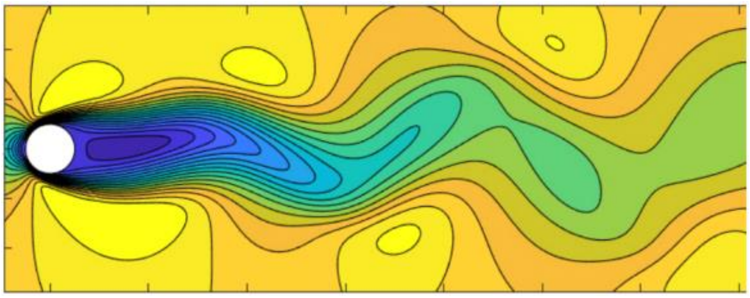
For example: data on two velocity components: $\boldsymbol{u}(\boldsymbol{z}_i, t) = \begin{pmatrix} u(\boldsymbol{z}_i, t) \\ v(\boldsymbol{z}_i, t) \end{pmatrix}$



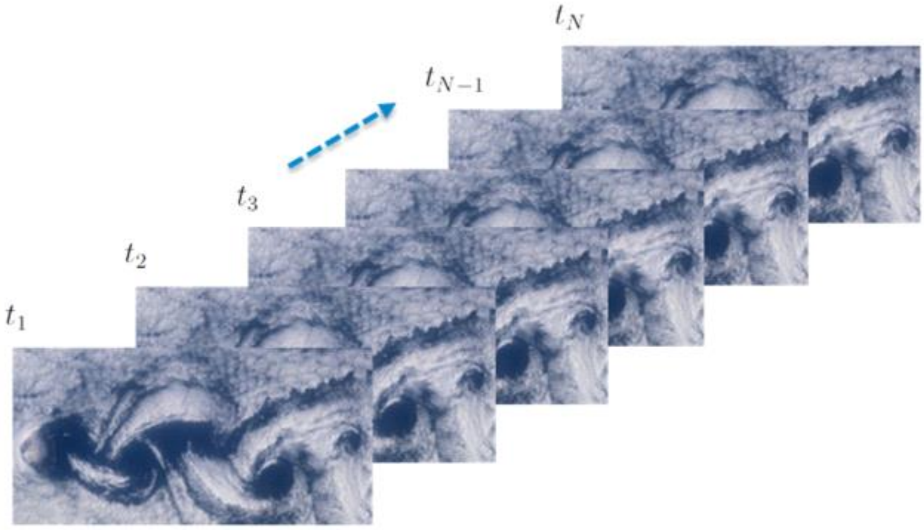
vectorised
perturbation
velocities

$$\boldsymbol{y} = \left(\begin{array}{c} u(\boldsymbol{z}_1, t) - \bar{u}(\boldsymbol{z}_1) \\ u(\boldsymbol{z}_2, t) - \bar{u}(\boldsymbol{z}_2) \\ \vdots \\ u(\boldsymbol{z}_p, t) - \bar{u}(\boldsymbol{z}_p) \\ \hline v(\boldsymbol{z}_1, t) - \bar{v}(\boldsymbol{z}_1) \\ v(\boldsymbol{z}_2, t) - \bar{v}(\boldsymbol{z}_2) \\ \vdots \\ v(\boldsymbol{z}_p, t) - \bar{v}(\boldsymbol{z}_p) \end{array}\right)$$

1.5 Coherent Structures From Data

- Next, suppose snapshots collected at times

$$t_1, \quad t_2, \quad t_3, \quad \ldots \quad t_{N-1}, \quad t_N$$



- Gives a series of snapshot data vectors

$$\boldsymbol{y}_1, \quad \boldsymbol{y}_2, \quad \boldsymbol{y}_3, \quad \cdots \quad \boldsymbol{y}_{N-1}, \quad \boldsymbol{y}_N$$

1.5 Coherent Structures From Data

- For data analysis create the **snapshot matrix**

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & y_{2N} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ y_{p1} & y_{p2} & \cdots & y_{pN} \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_N \\ \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{p \times N}$$

$$y_{ij} = \text{ velocity information at location } \boldsymbol{z}_i \text{ collected at time } t_j$$

1.6 Singular Value Decomposition

Definition 1. *Suppose that* $Y \in \mathbb{R}^{p\times N}$ *a snapshot matrix, let* $p > N$ *and assume that* $\text{rank}(Y) = N$. *The economy* **singular-value decomposition** *of* Y *is a decomposition into three matrices given by*

$$Y = U\Sigma W^\top$$

where

(i) $U \in \mathbb{R}^{p\times N}$ *satisfies* $U^\top U = I$*;*

(ii) $W \in \mathbb{R}^{N\times N}$ *satisfies* $W^\top W = I$*;*

(iii) $\Sigma \in \mathbb{R}^{N\times N}$ *is a diagonal matrix with positive entries.*

1.6 Singular Value Decomposition

The Relation to Coherent Structures

If we can compute the decomposition $Y = U\Sigma W^\top$ then:

1. The columns of U are the coherent structures:

$$U = \begin{pmatrix} \uparrow & & \uparrow \\ \mathbf{\Phi}_1 & \cdots & \mathbf{\Phi}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

   In fluid mechanics, these structures are called **POD modes**.

2. Diagonal entries of Σ are the **singular values** of Y:

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix}.$$

   These rank the importance of the coherent structures,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N.$$

3. The matrix W contains information required to find the linear combination of modes $\boldsymbol{\Phi}_i$ needed to construct each snapshot of the flow. For

$$Y = \begin{pmatrix} \uparrow & & \uparrow \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

the snapshot $\boldsymbol{y}_j$ sampled at time t_j can be written

$$\boldsymbol{y}_j = \sum_{i=1}^{N} \sigma_k \begin{pmatrix} \uparrow \\ \boldsymbol{\Phi}_k \\ \downarrow \end{pmatrix} w_{kj}$$

1.6 Singular Value Decomposition

- Can write the SVD as

$$Y = \sum_{k=1}^{N} \sigma_k \Phi_k \boldsymbol{w}_k^\top$$

- By using different numbers of terms in this sum, we can form different approximations to the snapshot matrix.

Theorem 1. *Suppose that we approximate Y by using $r \leq N$ coherent structures, i.e. letting*

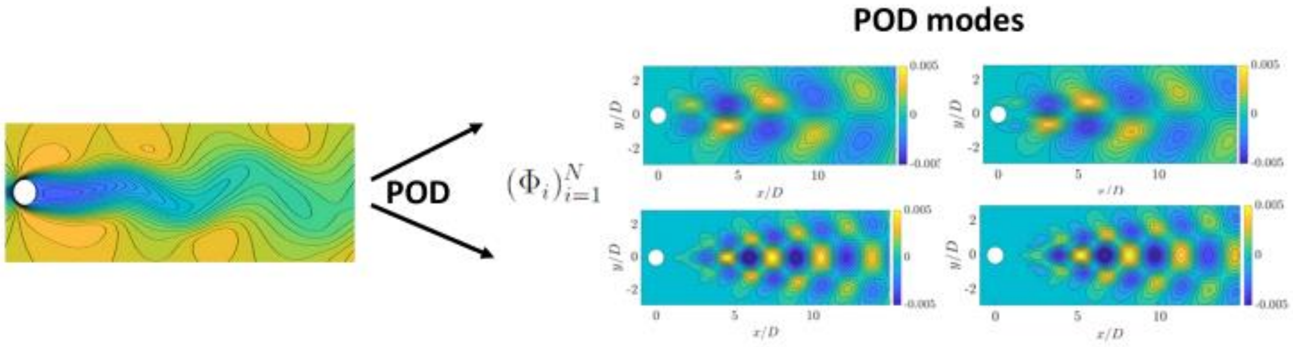
$$Y_r := \sum_{k=1}^{r} \sigma_k \Phi_k \boldsymbol{w}_k^\top .$$

Then this is the optimal rank-r approximation to the snapshot matrix Y in the sense that

$$\|Y - Y_r\|_F^2 = \min\left\{\|Y - B\|_F^2 : rank(B) = r\right\} = \sum_{k=r+1}^{N} \sigma_k^2$$

1.6 Some examples

- Unsteady flow past a circular cylinder $Re = 60$

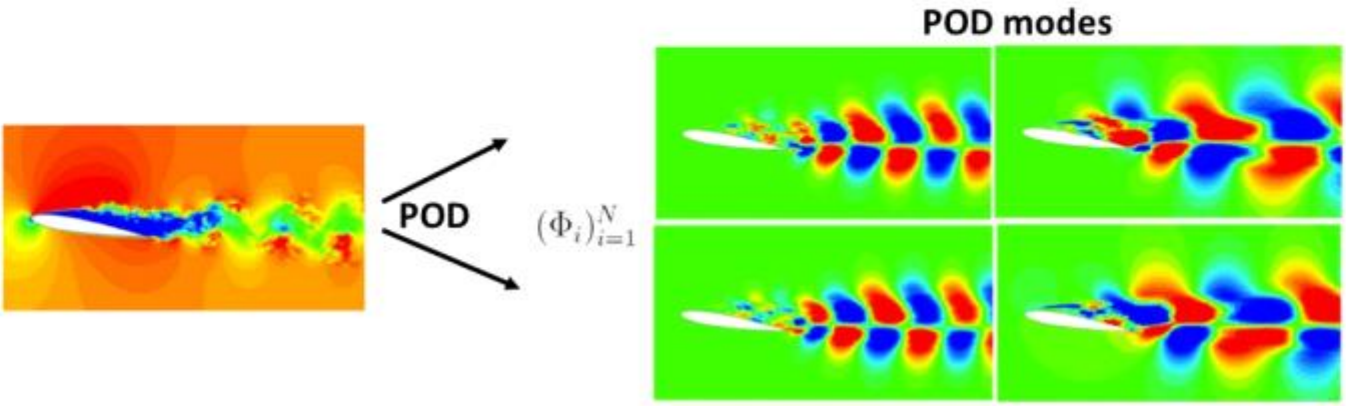


- **Question:** what percentage of the flow perturbation energy is described by a given number of modes?

$$E_r := \frac{\sum_{j=1}^{r} \sigma_j^2}{\sum_{j=1}^{N} \sigma_j^2}$$

1.6 Some examples

- Flow past an aerofoil at $Re = 23,000$

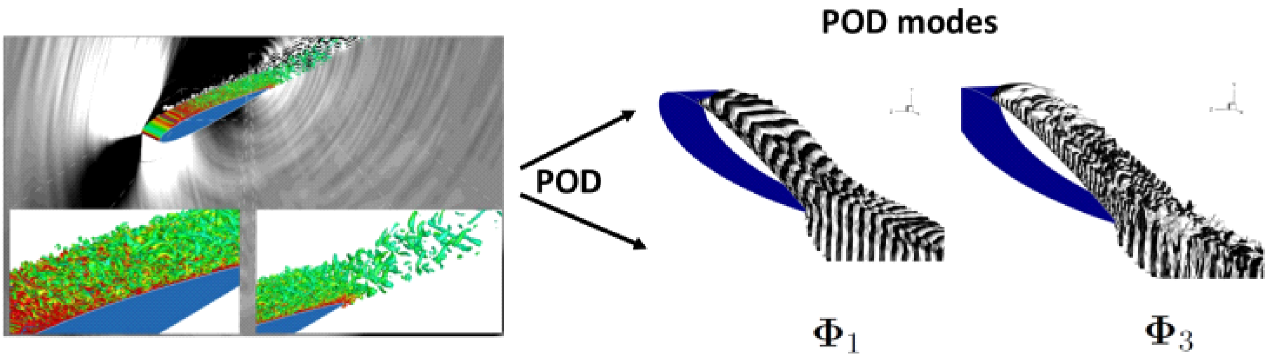


- **Question:** what percentage of the flow perturbation energy is described by a given number of modes?

$$E_r := \frac{\sum_{j=1}^{r} \sigma_j^2}{\sum_{j=1}^{N} \sigma_j^2}$$

1.6 Some examples

- Flow past an aerofoil at $Re = 408,000$



- **Question:** what percentage of the flow perturbation energy is described by a given number of modes?

$$E_r := \frac{\sum_{j=1}^{r} \sigma_j^2}{\sum_{j=1}^{N} \sigma_j^2}$$

1.7 Time dependent weights of POD modes

- Consider the case where snapshot matrix contains data about one velocity component, e.g.,

$$y_{ij} = u(\boldsymbol{z}_i, t_j) - \bar{u}(\boldsymbol{z}_i), \qquad i = 1, \ldots, p, \; j = 1, \ldots, N,$$

- Recall that the idea was to decompose

$$u(\boldsymbol{z}, t) - \bar{u}(\boldsymbol{z}_i) = x_1(t)\boldsymbol{\Phi}_1 + x_2(t)\boldsymbol{\Phi}_2 + \cdots + x_N(t)\boldsymbol{\Phi}_N$$

- **Question:** what are the time-dependent weights?

1.7 Time dependent weights of POD modes

$$u(\boldsymbol{z},t) - \bar{u}(\boldsymbol{z}_i) = x_1(t)\boldsymbol{\Phi}_1 + x_2(t)\boldsymbol{\Phi}_2 + \cdots + x_N(t)\boldsymbol{\Phi}_N$$

- **Question:** what are the time-dependent weights?

Proposition 1. *Suppose that* $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \ldots, \boldsymbol{\Phi}_N$ *are the POD modes calculated from the snapshot matrix* $Y \in \mathbb{R}^{p\times N}$. *Then the POD weights at sample times* $t_1 \leq t_2 \leq \cdots \leq t_N$ *of the first POD mode are*

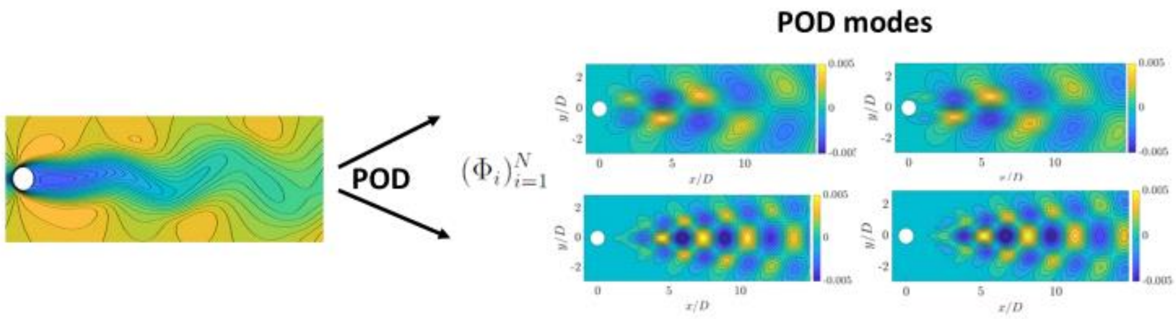
$$\boldsymbol{\Phi}_1^\top Y = \begin{pmatrix} x_1(t_1) & x_1(t_2) & \ldots & x_1(t_N) \end{pmatrix}$$

In general,

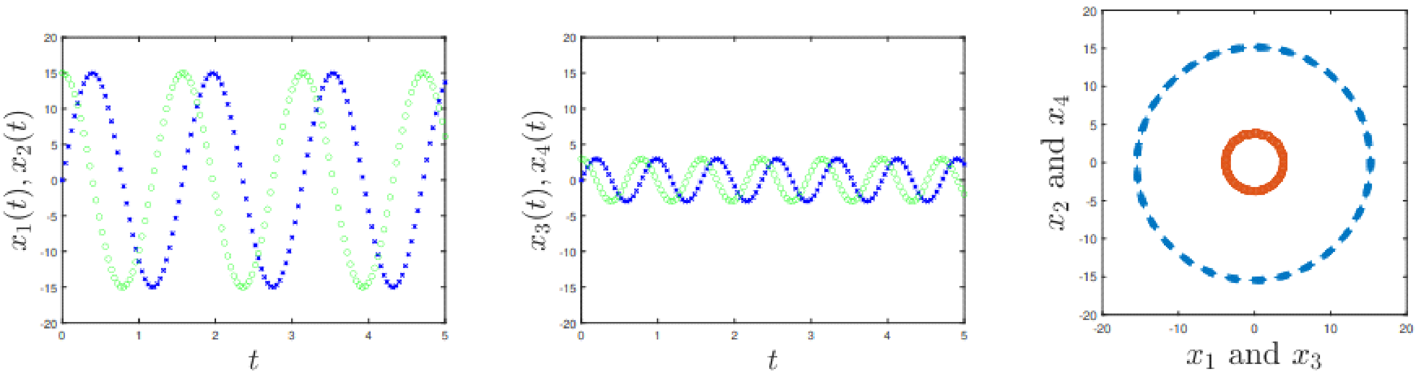
$$U^\top Y = \begin{pmatrix} x_1(t_1) & x_1(t_2) & \ldots & x_1(t_N) \\ x_2(t_1) & x_2(t_2) & \ldots & x_2(t_N) \\ \vdots & \vdots & & \vdots \\ x_N(t_1) & x_N(t_2) & \ldots & x_N(t_N) \end{pmatrix}$$

1.7 Some examples

- Unsteady flow past a circular cylinder at $Re = 60$

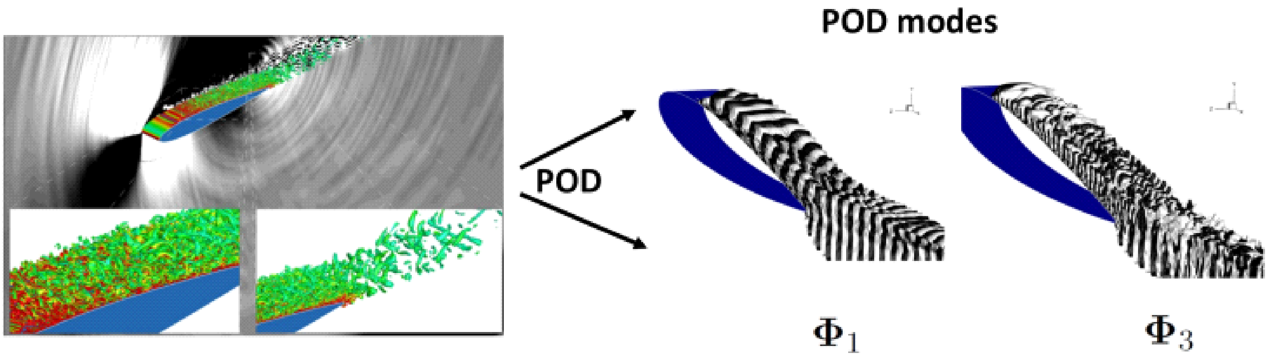


- POD weights

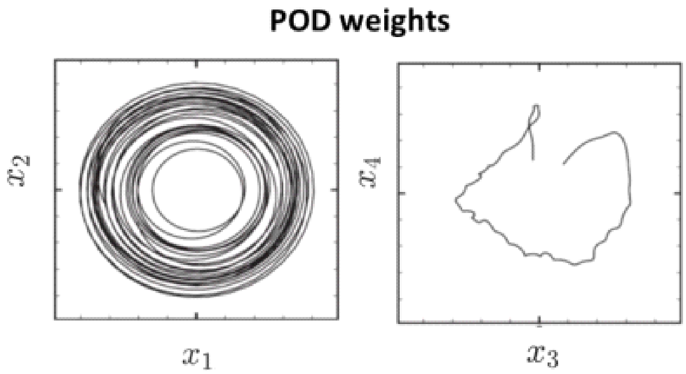


1.7 Some examples

- Flow past an aerofoil at $Re = 408,000$



- POD weights

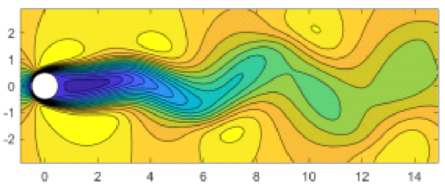


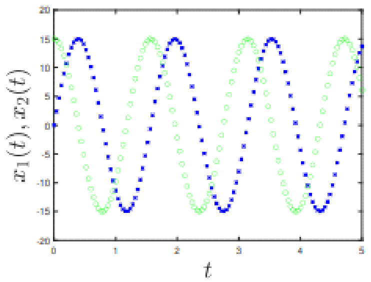
1.8 Dynamic Models from Data

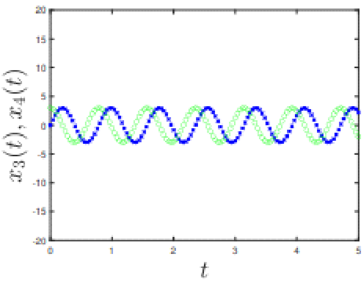
- We have seen that

  a. Coherent structures can be extracted from data
  b. Their weight sequences are temporally coherent

- **Idea:** fit a model to the time-dependent weight sequences







- Will look at a technique called **Dynamic Mode Decomposition**

1.8 Dynamic Models From Data

- Suppose we have collected flow snapshots

$$\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_N, \boldsymbol{y}_{N+1} \in \mathbb{R}^p$$

  at times

$$t_{j+1} = t_j + \Delta t, \qquad j = 1, \ldots, N.$$

- Idea is to model evolution over one timestep Δt

Question: If we look for a linear model, is it sensible to try to find a matrix such that

$$\boldsymbol{y}_{j+1} \approx \mathcal{A}\boldsymbol{y}_j, \qquad j = 1, \ldots, N.$$

where $\mathcal{A} \in \mathbb{R}^{p \times p}$?

1.8 Dynamic Models From Data

- To use SVD to reduce dimensions, first define two matrices

$$Y_B = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{p \times N}$$

$$Y_A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_2 & \boldsymbol{y}_3 & \cdots & \boldsymbol{y}_{N+1} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{p \times N}$$

1.8.1 The DMD Optimization Problem

- The original idea was to model snapshot evolution via

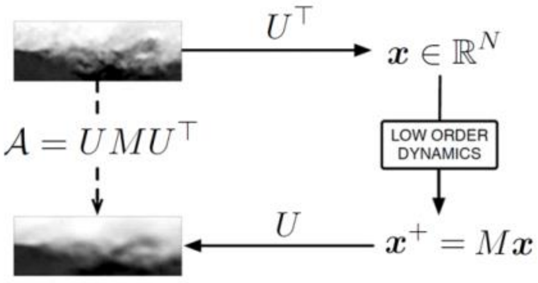
$$\boldsymbol{y}_{j+1} \approx \mathcal{A}\boldsymbol{y}_j, \qquad j = 1, \ldots, N.$$

- To test whether a given matrix is a good model, look at the residuals

$$\boldsymbol{r}_j = \boldsymbol{y}_{j+1} - \mathcal{A}\boldsymbol{y}_j, \qquad j = 1, \ldots, N.$$

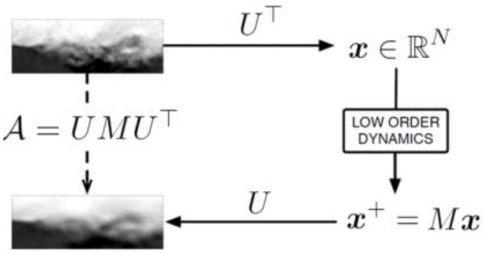
- Goodness of fit quantified by statistic

$$R = \sum_{j=1}^{N} \|\boldsymbol{r}_j\|^2 = \sum_{j=1}^{N} \left\|\boldsymbol{y}_{j+1} - UMU^\top \boldsymbol{y}_j\right\|^2$$



1.8.1 Dynamic Mode Decomposition (DMD)

- Minimising the fitting residuals gives an **optimal low-order linear model.**



Theorem 2 (Dynamic Mode Decomposition). *Let $Y \in \mathbb{R}^{p\times(N+1)}$ be a full-rank snapshot matrix and let $Y_A, Y_B \in \mathbb{R}^{p\times N}$ be the "after" and "before" snapshots. Then*

$$\operatorname{argmin}\left\{\sum_{j=1}^{N}\left\|\boldsymbol{y}_{j+1} - UMU^\top \boldsymbol{y}_j\right\|^2 : M \in \mathbb{R}^{N\times N}\right\} = U^\top Y_A W \Sigma^{-1}.$$

Consequently, an approximate model for the mode weights is $x(t_{k+1}) = Mx(t_k)$, where $M = U^\top Y_A W\Sigma^{-1}$.

1.8.1 Summary of Data-Driven Modelling

1. Start with a set $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{N+1} \in \mathbb{R}^p$ of snapshots of a fluid flow, sampled at times $t_1, t_2, \ldots, t_{N+1}$ with common time-step Δt.

2. Form the snapshot matrix

$$Y_B = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

3. Apply Proper Orthogonal Decomposition (POD) to Y_B to extract the POD modes $\boldsymbol{\Phi}_j$

$$Y_B = U\Sigma W^\top, \qquad U = \begin{pmatrix} \uparrow & & \uparrow \\ \boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_N \\ \downarrow & & \downarrow \end{pmatrix}$$

4. A reduced-order model of the flow be created with state

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^N.$$

A given value of $\boldsymbol{x} \in \mathbb{R}^N$ corresponds to a flow with velocity field

$$\boldsymbol{u}(\boldsymbol{z}) = \bar{\boldsymbol{u}}(\boldsymbol{z}) + \sum_{i=1}^{N} x_i(t)\boldsymbol{\Phi}_i(\boldsymbol{z})$$

5. To find an equation for the *dynamics* of the reduced-order state $\boldsymbol{x}$, create the "after" snapshot matrix

$$Y_A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_2 & \boldsymbol{y}_3 & \cdots & \boldsymbol{y}_{N+1} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

and solve the DMD optimization problem to obtain $M = U^\top Y_A W \Sigma^{-1}$.

6. The matrix $M \in \mathbb{R}^{N\times N}$ gives the optimal linear model describing the evolution of the flow contained in the collected snapshots. The reduced-order state $\boldsymbol{x} \in \mathbb{R}^N$ satisfies discrete-time dynamics

$$\boldsymbol{x}(t_{j+1}) = M\boldsymbol{x}(t_j), \qquad j \geq 0. \tag{5}$$

over a time-step of length $t_{j+1} - t_j = \Delta t$.

1.8.1 DMD Eigenvalues

- The **DMD eigenvalues** associated with the above modelling process are defined by

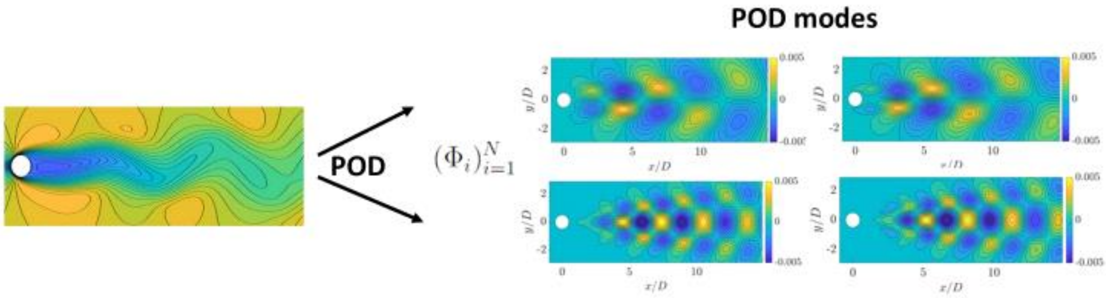
$$\lambda_i(A) = \frac{1}{\Delta t} \log \lambda_i(M), \qquad i = 1, \ldots, M$$

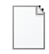
where $\lambda_i(M)$ are the eigenvalues of $M \in \mathbb{R}^{N\times N}$.

Question: why does this definition make sense?

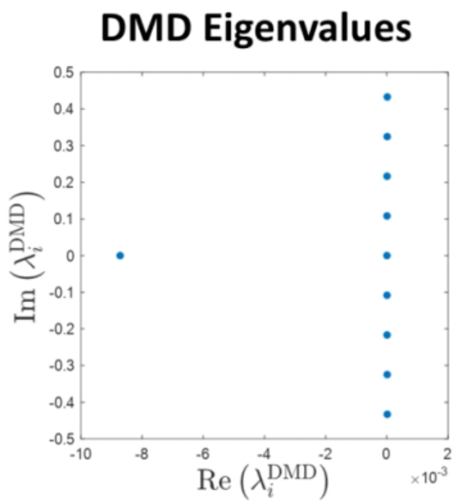
1.8.2 Some examples

- Unsteady flow past a circular cylinder at $Re = 60$



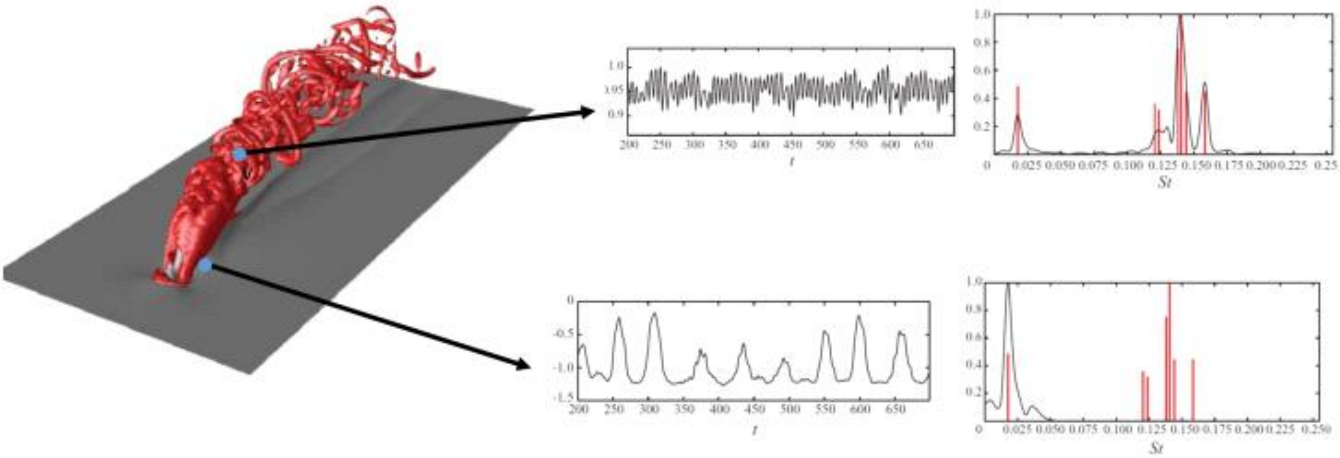


- DMD eigenvalues are on the imaginary axis



1.8.2 Some examples

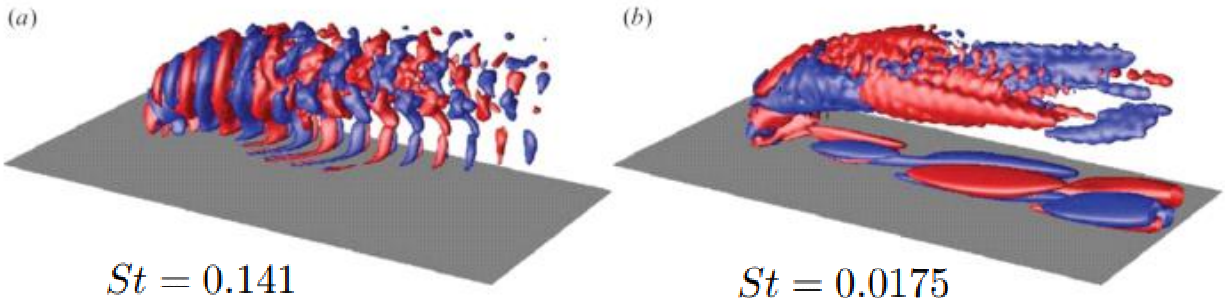
- A jet injected into a crossflow



[7]Taken from *C. Rowley et al., Spectral analysis of nonlinear flows, Journal of Fluid Mechanics, 2009.*

- Even this more complicated flow has clear dominant frequencies. The modes associated with these frequencies are

$$St = \frac{fD}{V_{\text{jet}}}$$



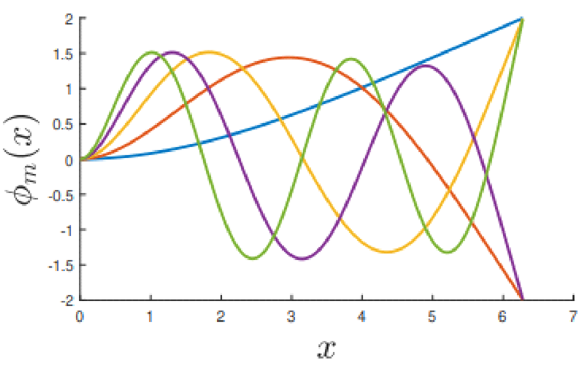
1.8.2 Some examples (Euler Bernoulli)

- Data-driven model of a cantilever Euler-Bernoulli beam

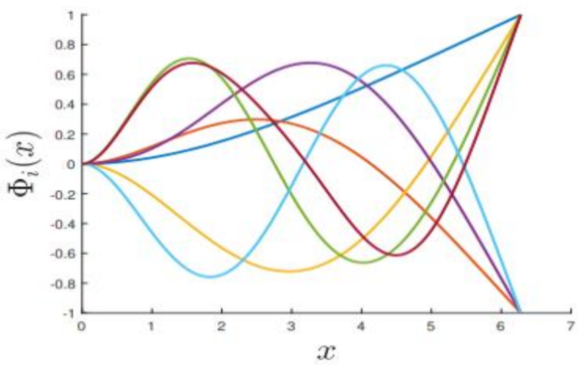
- From a simulation of the PDE, create a data matrix

$$Y = \begin{pmatrix} u(\boldsymbol{x}, t_1) & u(\boldsymbol{x}, t_1) & \cdots & u(\boldsymbol{x}, t_{N+1}) \\ u_t(\boldsymbol{x}, t_1) & u_t(\boldsymbol{x}, t_1) & \cdots & u_t(\boldsymbol{x}, t_{N+1}) \end{pmatrix} \in \mathbb{R}^{2p \times (N+1)}$$

- Taking the SVD gives mode shapes which match well with the dominant linear eigenfunctions



Eigenfunctions



POD modes

Also look at the DMD eigenvalues:

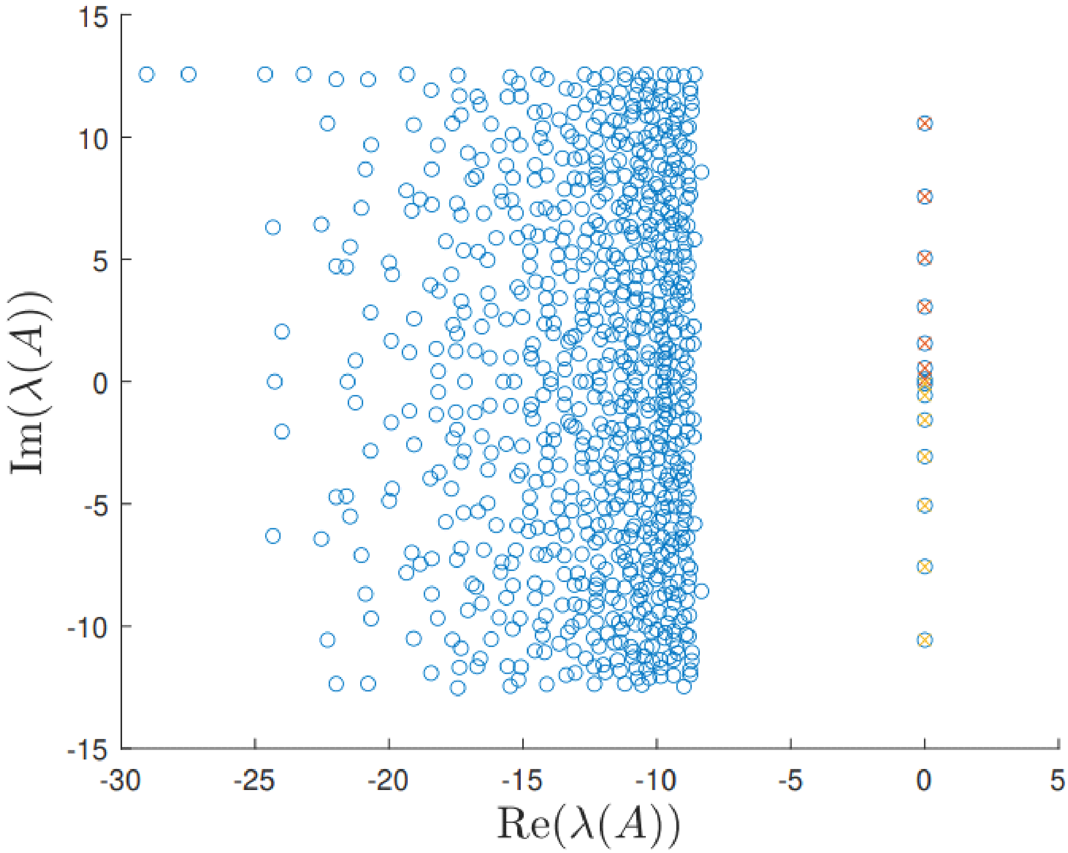


Figure 19: Eigenvalues of $A = \frac{1}{\Delta t}\log M$ shown in blue circles. Natural frequencies ω such that $\cosh(\sqrt{\omega}L)\cos(\sqrt{\omega}L) + 1 = 0$ are shown in red crosses.

Questions:

What are all the eigenvalues on the left of the plot?

Why are there so many?

Does it matter that they do not match with the natural frequencies?

1.8.2 Some examples (Euler Bernoulli)

- Also look at the DMD eigenvalues:

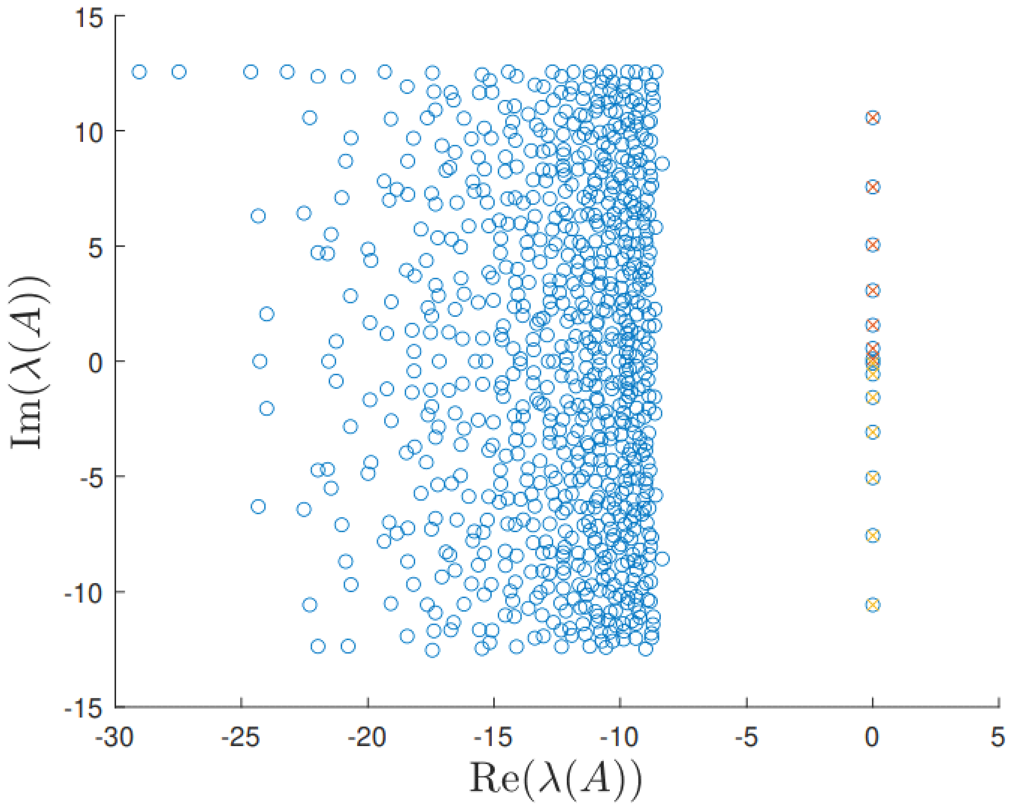


Figure 19: Eigenvalues of $A = \frac{1}{\Delta t} \log M$ shown in blue circles. Natural frequencies ω such that $\cosh\left(\sqrt{\omega} L\right) \cos\left(\sqrt{\omega} L\right) + 1 = 0$ are shown in red crosses.

Questions:
What are all the eigenvalues on the left of the plot?
Why are there so many?
Does it matter that they do not match with the natural frequencies?

1.9 DMD with control inputs

- Suppose we not only have snapshots

$$\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_N, \boldsymbol{y}_{N+1} \in \mathbb{R}^p$$

- But we also know these were collected at the same time as a series of applied control inputs

$$u_1 = u(t_1), u_2 = u(t_2), \ldots, u_{N+1} = u(t_{N+1}) \in \mathbb{R}$$

- Similar to DMD we seek a controlled model

$$\boldsymbol{y}_{j+1} \approx \mathcal{A}\boldsymbol{y}_j + \mathcal{B}u_j, \qquad j = 1, \ldots, N.$$

where $\mathcal{A} \in \mathbb{R}^{p\times p}$ and $\mathcal{B} \in \mathbb{R}^{p\times 1}$

- To find the state and input matrices, can use a similar idea to standard DMD:

$$Y_B = \left(\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_N \\ \downarrow & \downarrow & & \downarrow \\ \hline u_1 & u_2 & \ldots & u_N \end{array}\right) \in \mathbb{R}^{(p+1)\times N}$$

$$Y_A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{y}_2 & \boldsymbol{y}_3 & \cdots & \boldsymbol{y}_{N+1} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{p\times N}$$

- Taking the SVD of $Y_B = U\Sigma W^\top$ gives

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \qquad U_1 \in \mathbb{R}^{p\times N}, U_2 \in \mathbb{R}^{1\times N}$$

DMD model with Control

$$\boldsymbol{x}(t_{j+1}) = M\boldsymbol{x}(t_j) + \Gamma u(t_j), \qquad j \geq 0.$$

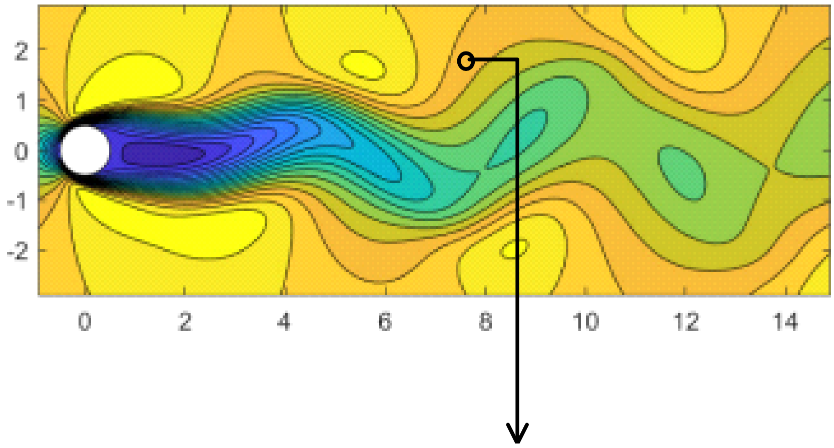
where

$$M = U_1^\top Y_A W \Sigma^{-1} \in \mathbb{R}^{N\times N} \qquad \text{and} \qquad \Gamma = U_1^\top Y_A W \Sigma^{-1} U_2^\top \in \mathbb{R}^{N\times 1}.$$

[8]see *Proctor et al. (2016) Dynamic Mode Decomposition with Control, SIAM Journal of Applied Dynamical Systems,* **15** *(1), 142–161.*

1.9 DMD with sensor measurements

- Suppose a sensor measurement can be taken from a flow



$$y(t) = u(\boldsymbol{z}_s, t)$$

- For example, one component of velocity is measured at a single location in the flow domain

  Using the assumed series decomposition

$$y(t) = u(\boldsymbol{z}_s, t) = \bar{u}(\boldsymbol{z}_s) + \sum_{i=1}^{N} x_i(t)\Phi_i(\boldsymbol{z}_s)$$

1.9 Summary

- Have shown how data ensembles can be used to create finite-dimensional control systems of the form

$$\boldsymbol{x}(t_{j+1}) = M\boldsymbol{x}(t_j) + \Gamma u_j$$
$$y_j = C\boldsymbol{x}(t_j)$$

- The link to an original state, typically the solution to a PDE is

$$\boldsymbol{u}(\boldsymbol{z}, t) = \bar{\boldsymbol{u}} + \sum_{i=1}^{N} x_i(t)\boldsymbol{\Phi}_i$$

- Have seen that solutions agree well with known analytical solutions from linear beam theory.

- The power of the technique is that it can be applied generally to any data set drawn from a dynamical system.