2. Stability

- In the first section of lectures we have seen how to create finite dimensional discrete time models

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k, \qquad \boldsymbol{x}_k \in \mathbb{R}^n, \boldsymbol{u}_k \in \mathbb{R}^m$$

  or continuous time models

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

- These can be used to give reduced-order approximations to controlled dynamical systems

- Recalling that the state $\boldsymbol{x}(t)$ describes **perturbations**, the standard aim of controller is to use $\boldsymbol{u}(t)$ to stabilize the system, i.e., ensure

$$\lim_{t\to\infty} x(t) = 0.$$

- Purpose of this section is to derive checkable conditions for stability.

2.1 Discrete Time Stability

In this section we consider discrete time systems

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$$
$$x_0 \in \mathbb{R}^n$$

where we assume that

(i) $\boldsymbol{x}_k \in \mathbb{R}^n$ is the system state

(ii) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous

(iii) and $f(0) = 0$ is an equilibrium point.

- Will give a very brief introduction to **Lyapunov stability theory.**

- The results we derive apply to both linear and nonlinear systems and give a systematic method of determining system stability.

2.1 Lyapunov Stability

Definition 2. *The equilibrium point $x = 0$ of the system (7) is (Lyapunov) stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|\boldsymbol{x}_0\| \leq \delta \Longrightarrow \|\boldsymbol{x}_k\| \leq \epsilon, \quad \textit{for all}\ k \geq 0.$$

2.1 Asymptotic Stability

Definition 3. *The equilibrium point $x = 0$ of the system (7) is asymptotically stable if i) it is Lyapunov stable and ii) there exists $\delta > 0$ such that*

$$\|\boldsymbol{x}_0\| \leq \delta \Longrightarrow \lim_{k \to \infty} \boldsymbol{x}_k = 0.$$

The equilibrium is said to be Globally Asymptotically Stable (GAS) is $\lim_{k\to\infty} \boldsymbol{x}_k = 0$ holds for any $\boldsymbol{x}_k \in \mathbb{R}^n$.

2.2 Linear Stability

Example 9. *Suppose that* $\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$ *is linear with* $f(\boldsymbol{x}) = A\boldsymbol{x}$ *for some matrix* $A \in \mathbb{R}^{n\times n}$*. Then the equilibrium point* $x = 0$ *is stable (and GAS) if and only if*

$$\sigma(A) \subset \mathbb{D},$$

where $\sigma(A) \subset \mathbb{C}$ *is the set of eigenvalues of* A*.*

2.3 Nonlinear Stability

- **Question:** There are nice checkable conditions for linear stability. What results are available for nonlinear systems?

- **Idea:** Suppose that $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$ and that $\boldsymbol{x}_0 = \boldsymbol{v}$ with

$$A\boldsymbol{v} = \lambda\boldsymbol{v}, \quad \text{and} \quad |\lambda| < 1$$

Then…

2.3 Nonlinear Stability

- initial idea was to use the norm decrease condition

$$\|f(\boldsymbol{x})\|_2 - \|\boldsymbol{x}\|_2 < 0, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

As a way of checking stability of the nonlinear system $\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$

Example 10. *Consider the system* $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$ *with*

$$A = \begin{pmatrix} 0.99 & 0.99 \\ 0 & 0.99 \end{pmatrix}.$$

Show that the system is globally asymptotically stable. Show further that the norm-decrease condition $\|A\boldsymbol{x}_k\| - \|\boldsymbol{x}_k\| < 0$ *does not hold along all trajectories of the system.*

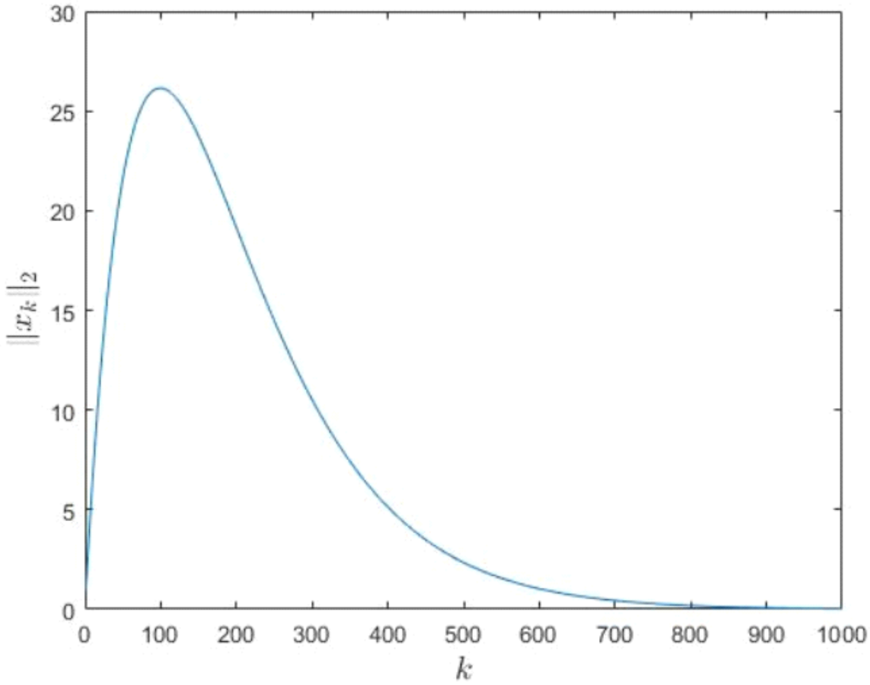


Figure 20: Trajectory $x_{k+1} = Ax_k$ from $x_0 = (1/\sqrt{2}, 1/\sqrt{2})$.

2.4 Lyapunov Functions

- Have just shown that the naïve approach to extending linear to nonlinear stability fails. The following result introduces the notion of a **Lyapunov Function** to fix this problem

Theorem 3. *Let $0 \in D \subset \mathbb{R}^n$ be a domain containing the equilibrium point $x = 0$ of the system $\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$. Suppose there exists a continuous function $V : D \to \mathbb{R}$ which satisfies the following three conditions:*

i) $V(0) = 0$; and $V(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in D$ with $\boldsymbol{x} \neq 0$;

ii) There exist constants α_1, α_2 such that

$$\alpha_1 \|\boldsymbol{x}\|_2 \leq V(\boldsymbol{x}) \leq \alpha_2 \|\boldsymbol{x}\|_2, \qquad \boldsymbol{x} \in D;$$

iii) $V(f(\boldsymbol{x})) - V(\boldsymbol{x}) \leq 0$ for any $\boldsymbol{x} \in D$.

Then the equilibrium point $\boldsymbol{x} = 0$ is Lyapunov stable.

2.4 Lyapunov Functions

- A small addition gives conditions for asymptotic stability.

Theorem 4. *Suppose that the conditions of Theorem 3 hold with* $D = \mathbb{R}^n$ *and the third condition replaced by*

$$V(f(\boldsymbol{x})) - V(\boldsymbol{x}) < 0, \qquad 0 \neq \boldsymbol{x} \in \mathbb{R}^n.$$

Then the equilibrium point $\boldsymbol{x} = 0$ *of the system* $\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k)$ *is globally asymptotically stable.*

2.4 K-functions

- With a view towards constructing Lyapunov functions, the upper and lower bound condition

$$\alpha_1 \|\boldsymbol{x}\|_2 \leq V(\boldsymbol{x}) \leq \alpha_2 \|\boldsymbol{x}\|_2$$

appears to place specific (linear) growth rate on the Lyapunov function. This is not necessary.

> **Definition 4.** *A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of type $\mathcal{K}_\infty$ if it is strictly increasing with $\alpha(x) \to \infty$ as $x \to \infty$ and satisfies $\alpha(0) = 0$.*

- The growth condition can be replaced by the more general:

> **Remark 1.** *In Theorem 4, the condition ii) placing upper and lower bounds on V can be replaced by the following: there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*
>
> $$\alpha_1(\|\boldsymbol{x}\|_2) \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|_2), \qquad x \in \mathbb{R}^n$$

2.5 Lyapunov Stability for Linear Systems

Definition 5. *A real symmetric matrix $P \in \mathbb{R}^{n \times n}$ is said to be strictly positive definite if*

$$\boldsymbol{x}^\top P \boldsymbol{x} > 0, \qquad 0 \neq \boldsymbol{x} \in \mathbb{R}^n$$

and we write $P \succ 0$ if this is the case. A negative definite matrix $P \prec 0$ is definied in an analgous manner.

- Recalling the positivity condition of Lyapunov functions, a possible class of Lyapunov functions is given by

$$V(\boldsymbol{x}) = \boldsymbol{x}^\top P \boldsymbol{x}, \qquad (\text{for any } P \succ 0)$$

- For real, symmetric, matrices positive definiteness can be checked by looking at eigenvalues

$$\lambda_{\min}(P) = \min_{0 \neq \boldsymbol{x} \in \mathbb{R}^n} \frac{\boldsymbol{x}^\top P \boldsymbol{x}}{\|\boldsymbol{x}\|^2} \quad \text{and} \quad \lambda_{\max}(P) = \max_{0 \neq \boldsymbol{x} \in \mathbb{R}^n} \frac{\boldsymbol{x}^\top P \boldsymbol{x}}{\|\boldsymbol{x}\|^2}$$

2.5 Lyapunov Stability for Linear Systems

- Quadratic Lyapunov functions can be used to characterise stability of linear systems

Example 11. *Consider* $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$ *with* $\boldsymbol{x}_0 \in \mathbb{R}^n$. *Show that the system is globally asymptotically stable if there exists a matrix* $P \succ 0$ *such that*

$$A^\top PA - P \prec 0.$$

2.5 Lyapunov Stability for Linear Systems

- Look again at the linear system with large transient growth

Example 12. *Consider the linear system* $\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$ *with*

$$A = \begin{pmatrix} 0.99 & 0.99 \\ 0 & 0.99 \end{pmatrix}.$$

Find a Lyapunov function of the form $V(\boldsymbol{x}) = \boldsymbol{x}^\top P\boldsymbol{x}$ *which implies that the system is GAS.*

2.6 Level Sets

- The previous calculation was a painful way to check stability of a linear system!

- However, an advantage of the Lyapunov approach is that it gives information about the geometry of trajectories of the system.

- To explain this recall that we have looked at **level sets**

$$\Omega_b := \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \leq b\}$$

- For the system in the previous example

$$V(x) = \boldsymbol{x}^\top P \boldsymbol{x}, \qquad P = \begin{pmatrix} 1 & (1-a^2)^{-1} \\ (1-a^2)^{-1} & 3(1-a^2)^{-2} \end{pmatrix}$$

from which the level sets can be computed…

2.6 Level Sets

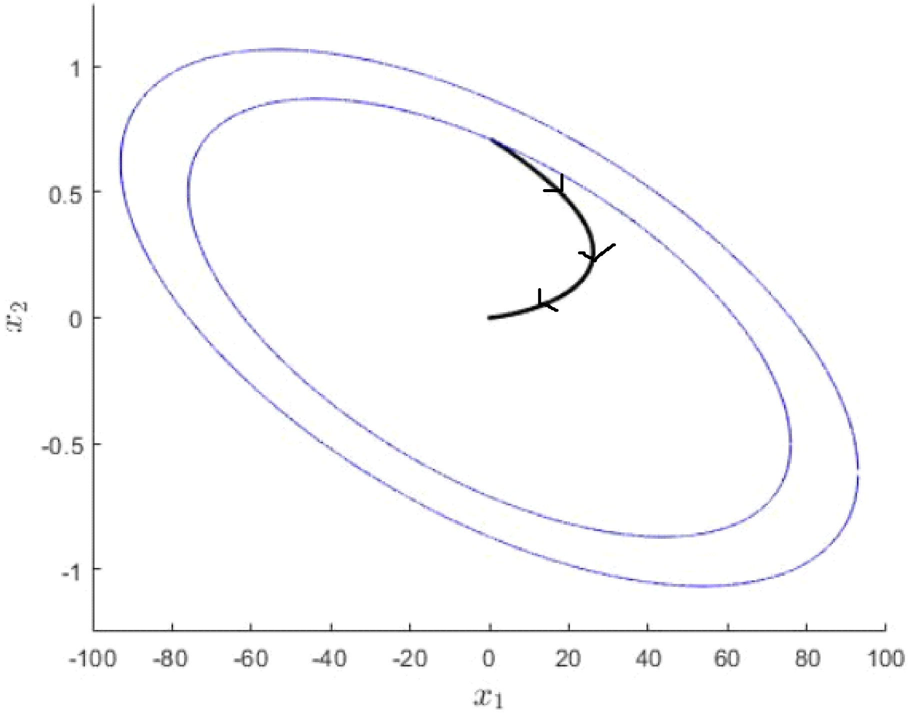


Figure 21: Level sets Ω_{1920} and Ω_{2880} for the Lyapunov function $V(x) = x^\top Px$. Also shown is the trajectory $(x_k)_{k\geq 0}$ of the system $x_{k+1} = Ax_k$ with $x_0 = (1/\sqrt{2}, 1/\sqrt{2})$ with $A = \left(\begin{smallmatrix} 0.99 & 0.99 \\ 0 & 0.99 \end{smallmatrix}\right)$.

$$V(x) = \boldsymbol{x}^\top P\boldsymbol{x}, \qquad P = \begin{pmatrix} 1 & (1-a^2)^{-1} \\ (1-a^2)^{-1} & 3(1-a^2)^{-2} \end{pmatrix}$$

$$\Omega_b := \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \leq b\}$$

2.7 Continuous Systems

- Now consider continuous time systems

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t))$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$$

- Assume that

(i) $\boldsymbol{x}(t)$ is the system state at time $t \geq 0$

(ii) $f : D \to \mathbb{R}^n$ is a Lipschitz continuous

(iii) $f(0) = 0$ is an equilibrium point.

2.7 Stability (Continuous Systems)

- There are analogous definitions for stability:

Definition 6. *The equilibrium point* $\boldsymbol{x} = 0$ *of the system* $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t))$ *is:*

i) Stable if for any $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\|\boldsymbol{x}_0\| \leq \delta \Longrightarrow \|\boldsymbol{x}(t)\| \leq \epsilon, \quad \text{for all } t \geq 0.$$

ii) Locally asymptotically stable if there exists $\delta > 0$ *such that*

$$\|\boldsymbol{x}(0)\| < \delta \Longrightarrow \lim_{t \to \infty} \boldsymbol{x}(t) = 0.$$

iii) Globally asymptotically stable if for any $\boldsymbol{x}_0 \in \mathbb{R}^n$*, it follows that* $\boldsymbol{x}(t) \to 0$ *as* $t \to \infty$*.*

2.7 Lyapunov Stability (Continuous Systems)

- There are analogous Lyapunov conditions to verify stability:

Theorem 5. *Suppose that $\boldsymbol{x} = 0$ is an equilibrium point for $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Suppose that a continuously differentiable function $V : D \to \mathbb{R}$ exists which satisfies*

i) $V(0) = 0$; *and* $V(\boldsymbol{x}) > 0$, *for every* $0 \neq \boldsymbol{x} \in D$;

ii)

$$(\nabla V)(\boldsymbol{x}) \cdot f(\boldsymbol{x}) \leq 0, \qquad \boldsymbol{x} \in D.$$

Then $\boldsymbol{x} = 0$ *is stable equilibrium point for* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.

Question: Why is condition ii) the natural extension of the contractive condition for discrete time systems?

2.7 Asymptotic Stability (Continuous Systems)

- There are analogous Lyapunov conditions to verify stability:

Theorem 6. *Suppose that* $\boldsymbol{x} = 0$ *is an equilibrium point for* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. *Suppose that a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$ *exists which satisfies*

i) $V(0) = 0$; *and* $V(\boldsymbol{x}) > 0$, *for every* $0 \neq \boldsymbol{x} \in \mathbb{R}^n$;

ii) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that*

$$\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|), \qquad \boldsymbol{x} \in \mathbb{R}^n;$$

iii)

$$(\nabla V)(\boldsymbol{x}) \cdot f(\boldsymbol{x}) < 0, \qquad 0 \neq \boldsymbol{x} \in \mathbb{R}^n.$$

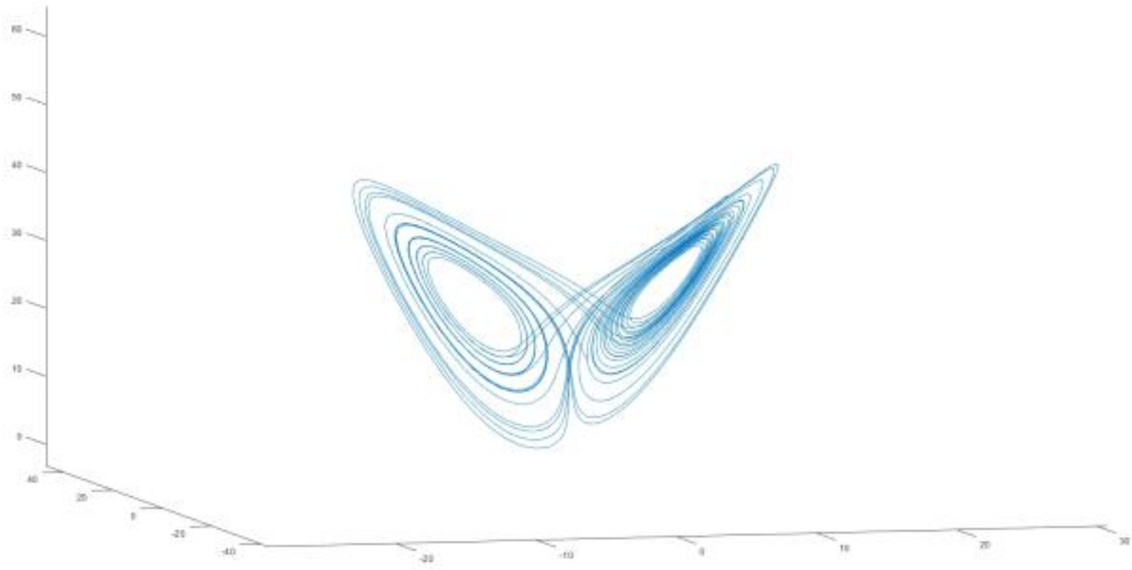
Then $\boldsymbol{x} = 0$ *is a globally asymptotically stable equilibrium point.*

2.8 Example: the Lorenz system

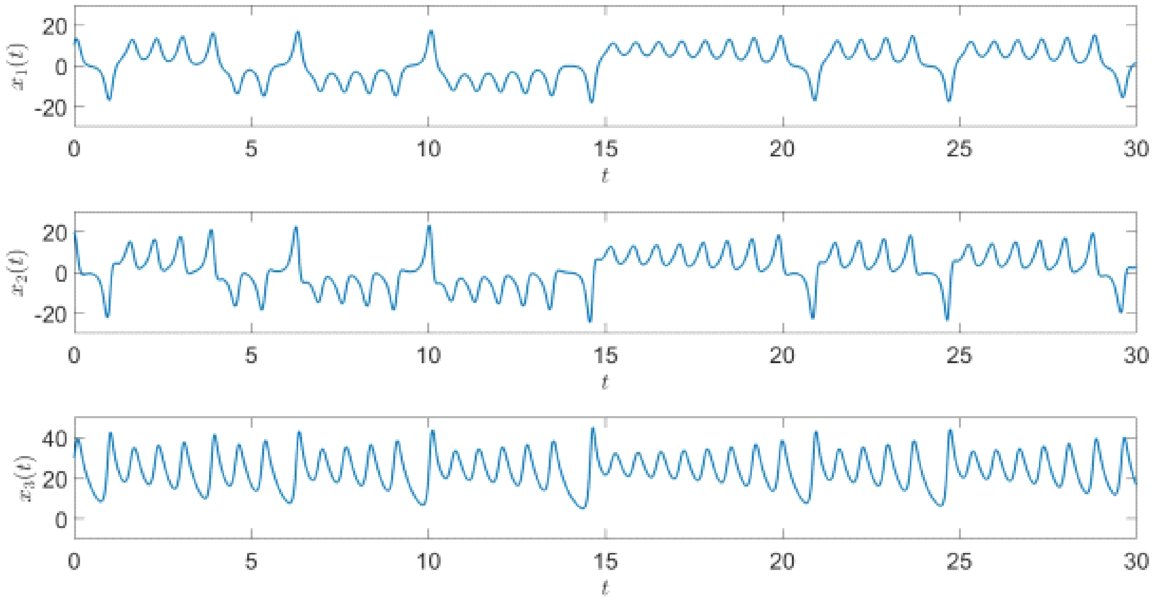
- Consider the Lorenz system

$$\dot{x}_1 = \sigma x_2 - \sigma x_1$$
$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3$$
$$\dot{x}_3 = -\beta x_3 + x_1 x_2.$$

- For certain parameter values, this famously exhibits chaotic motion



(a) 3D view



(b) Lorenz states

Figure 22: The Lorenz attractor for parameters $(\sigma, \rho, \beta) = (10, 28, 8/3)$.

2.8 Example: the Lorenz system

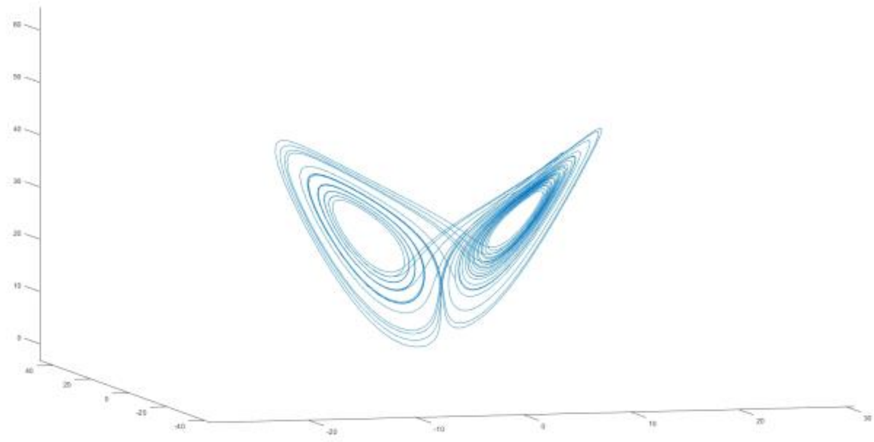
$$\dot{x}_1 = \sigma x_2 - \sigma x_1$$
$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3$$
$$\dot{x}_3 = -\beta x_3 + x_1 x_2.$$

Example 13. *Consider the Lorenz system with parameters* (σ, ρ, β). *Show that* $\boldsymbol{x} = 0 \in \mathbb{R}^3$ *is globally asymptotically stable if* $0 < \rho < 1$.

2.8 Example: the Lorenz System

- The previous example showed that it is possible to prove global stability of nonlinear systems.

- However, this example misses the point for the Lorenz system in so far as its interesting behaviour **is not** for the parameter values where we can prove global stability



(a) 3D view

- To prove a result which says something interesting requires us to be able to say sometime about the attractor.

2.9 Estimating Attractors

- The following result says that if we can verify the Lyapunov decay condition outside a ball, then the system's state is bounded.

Theorem 7. *Consider the ODE* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. *Suppose that* $V : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable, satisfies conditions i) and ii) from Theorem 6 and, in addition, there exists* $R > 0$ *such that*

$$(\nabla V)(\boldsymbol{x}) \cdot f(\boldsymbol{x}) < 0, \qquad \|\boldsymbol{x}\| \geq R.$$

Then there exists a ball $B_Q = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq Q\}$ *such that the solution* $(\boldsymbol{x}(t))_{t\geq 0}$ *enters* B_Q *and remains in this ball for sufficiently large times* t.

Example 14. *Show that there exits a ball* B_Q *such that any solution* $\boldsymbol{x}(t)$ *to Lorenz system must enter* B_Q *for sufficiently large* $t \geq 0$.

2.10 Lasalle's Invariance Theorem

- The previous result showed that the decay condition could still be useful if it is only known in a certain domain.
- The following result is a second way in which partial decay information can be used.

Theorem 8 (LaSalle's invariance theorem). *Suppose that $\boldsymbol{x} = 0$ is an equilibrium point for $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Suppose that there exists $V : \mathbb{R}^n \to \mathbb{R}$ satisfying conditions i) and ii) of Theorem 6. Suppose that*

$$(\nabla V)(\boldsymbol{x}) \cdot f(\boldsymbol{x}) \leq 0, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

and let $\Sigma = \{\boldsymbol{x} : (\nabla V)(\boldsymbol{x}) \cdot f(\boldsymbol{x}) = 0\}$. If it is the case that the only solution to (10) that can remain in Σ is the zero solution $\boldsymbol{x}(t) \equiv 0$, then the equilibrium point $\boldsymbol{x} = 0$ is globally asymptotically stable.

2.10 An Example

- The following example shows how the invariance theorem can show global stability even if the decay condition does not hold everywhere.

Example 15. *Consider the linear system* $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ *and* $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$. *Suppose that there exists a positive definite matrix* $P \succ 0$ *and a matrix* $C \in \mathbb{R}^{1\times n}$ *such that*

$$PA + A^\top P = -C^\top C$$

Suppose that, in addition, (A, C) *satisfy the observability property*

$$\int_0^\infty \|Ce^{tA}\boldsymbol{x}_0\|^2 dt \geq k\|\boldsymbol{x}_0\|^2, \qquad \boldsymbol{x}_0 \in \mathbb{R}^n$$

for some constant $k > 0$. *Under these conditions, show that the system is globally asymptotically stable.*