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MODCONFLEX
Modelling and control of flexible
structures interacting with fluids

Port Hamiltonian systems from analysis to numerics

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Introduction to lecturer and material

- The lecturer

What will we cover?

1. Dirac structures on finite-dimensional spaces.
   1.1 General definition and properties
   1.2 Defining continuous- and discrete-time systems via a Dirac structure; ODE's, DAE's
2. Dirac structures on infinite-dimensional spaces.
   2.1 Gently introduction
   2.2 Class of Dirac structures
   2.3 Link to operators and PDE's.
3. Restricting a Dirac structure on infinite-dimensional spaces to finite-dimensional space (numerics).
4. Existence of solution of pH-PDE's.
   4.1 Homogeneous
   4.2 Inhomogeneous
   4.3 Transfer functions

Port Hamiltonian systems from analysis to numerics

Dirac structures

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Dirac structures, general

Let $\mathcal{E}$ and $\mathcal{F}$ be real (complex) two linear spaces with a bilinear product

$$\langle f \mid e \rangle \in \mathbb{R} \text{ (or } \mathbb{C}).$$

We assume that this product is non-degenerated, that is

$$\langle f \mid e \rangle = 0 \quad \forall e \in \mathcal{E} \Rightarrow f = 0,$$
$$\langle f \mid e \rangle = 0 \quad \forall f \in \mathcal{F} \Rightarrow e = 0.$$

$\mathcal{E}$ is called the effort space and $\mathcal{F}$ is the flow space. The bond space $\mathcal{B}$ is defined as $\mathcal{F} \times \mathcal{E}$.

Dirac structures, general

On the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ we define the symmetrised pairing

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{B}} = \langle f_2 \mid e_1 \rangle + \langle f_1 \mid e_2 \rangle.$$

For $V \subseteq \mathcal{B}$ we define

$$V^{\perp} = \left\{ \begin{pmatrix} f_1 \\ e_1 \end{pmatrix} \in \mathcal{B} \mid \left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{B}} = 0 \text{ for all } \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \in V \right\}.$$

Definition

The linear subspace $\mathcal{D}$ of $\mathcal{B}$ is a Dirac structure if $\mathcal{D}^{\perp} = \mathcal{D}$. □

Dirac structures, general properties

If $\mathcal{D}$ is a Dirac structure, then

$$\langle f \mid e \rangle = 0 \text{ for all } \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D}.$$

This has (may have) the interpretation of power conservation, see later.

Dirac structures, finite-dimensional

For finite-dimensional spaces, the following gives a very useful characterisation of a Dirac structure.

Lemma

For $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$ with

$$\langle f \mid e \rangle = f^\top e$$

we have that $\mathcal{D}$ is a Dirac structure if and only if there exists two $n \times n$ matrices F and E, such that

1. $\mathcal{D} = \operatorname{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$*;*
2. *The matrix $\left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ has full rank (rank equals n);*
3. $F^\top E + E^\top F = 0$*, or in other words $F^\top E$ is skew-adjoint (anti-symmetric).*

Question Formulate a similar result if $\langle f \mid e \rangle = f^\top Q e$. Conditions on Q?

Proof of Lemma

Assume that $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a Dirac structure. Then

- It is a linear subspace, so there exist matrices F and E of size $(n \times m)$ such that $\mathcal{D} = \operatorname{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ is of full rank (rank equals m).
- The relation $\left(\begin{smallmatrix} f_2 \\ e_2 \end{smallmatrix} \right) \perp \operatorname{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ is a linear equation with $2n$ unknown and m conditions. Hence the dimension of the solution set is $2n - m$-dimensional. However, since the solution set equals $\mathcal{D}$ we have $2n - m = m$. Thus $m = n$.
- The equality $\langle f \mid e \rangle = 0$ is equivalent to $\ell^\top F^\top E \ell = 0$ for all $\ell \in \mathbb{R}^n$. Thus $\ell^\top \left[F^\top E + E^\top F \right] \ell = 0$. Since $F^\top E + E^\top F$ is symmetric, we conclude that $F^\top E + E^\top F = 0$.

Dirac structures, finite-dimensional

Proof of Lemma, continued.

Let $\mathcal{D} = \operatorname{ran}\left(\begin{smallmatrix} F \\ E \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} F \\ E \end{smallmatrix}\right)$ a $2n \times n$ matrix of rank n, and with $F^\top E + E^\top F = 0$. We have to show that $\mathcal{D}$ is a Dirac structure.

- For $\left(\begin{smallmatrix} f_2 \\ e_2 \end{smallmatrix}\right) \in \mathcal{D}$ we have for any $\left(\begin{smallmatrix} f_1 \\ e_1 \end{smallmatrix}\right) \in \mathcal{D}$ that

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{B}} = \langle f_2 \mid e_1 \rangle + \langle f_1 \mid e_2 \rangle$$
$$= \ell_2^\top F^\top E \ell_1 + \ell_1^\top F^\top E \ell_2$$
$$= \ell_2^\top F^\top E \ell_1 + \ell_2^\top E^\top F \ell_1 = 0.$$

  Thus $\left(\begin{smallmatrix} f_2 \\ e_2 \end{smallmatrix}\right) \in \mathcal{D}^\perp$, and so $\mathcal{D} \subseteq \mathcal{D}^\perp$.

- By construction $\dim(\mathcal{D}^\perp) = 2n - n$ (dimension space minus number of conditions) $= n = \dim(\mathcal{D})$.
  Combined with $\mathcal{D} \subseteq \mathcal{D}^\perp$, we find that $\mathcal{D} = \mathcal{D}^\perp$.

Dirac structures, finite-dimensional

We assume the finite-dimensional case, i.e., $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$ with $\langle f \mid e \rangle = f^\top e$.
We have seen that every Dirac structure can be written as $\mathcal{D} = \operatorname{ran} \begin{pmatrix} F \\ E \end{pmatrix}$ with $\begin{pmatrix} F \\ E \end{pmatrix}$ a $2n \times n$ matrix of rank n, and with $F^\top E + E^\top F = 0$. This is known as the image representation.

Lemma

Let the Dirac structure on $\mathbb{R}^n \times \mathbb{R}^n$ be given as $\mathcal{D} = \operatorname{ran} \begin{pmatrix} F \\ E \end{pmatrix}$ with the above condition on E, F. Then $\mathcal{D}$ has the kernel representation

$$\mathcal{D} = \ker \begin{pmatrix} E^\top & F^\top \end{pmatrix}.$$

Question Under which condition(s) is $\mathcal{D} = \ker \begin{pmatrix} E_1 & F_1 \end{pmatrix}$ a Dirac structure? Furthermore, what is its image representation?

Dirac structures, finite-dimensional

Proof of the Lemma

For $\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{D}$ we have

$$\begin{pmatrix} E^\top & F^\top \end{pmatrix} \begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} E^\top & F^\top \end{pmatrix} \begin{pmatrix} F \\ E \end{pmatrix} \ell = 0.$$

Hence $\mathcal{D} \subseteq \ker \begin{pmatrix} E^\top & F^\top \end{pmatrix}$.

By checking dimensions, we find that these sets are equal. $\square$

Dirac structures, finite-dimensional

We assume the finite-dimensional case, i.e., $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$ with $\langle f \mid e \rangle = f^\top e$. We have the following alternative characterisation of a Dirac structure.

Lemma

$\mathcal{D} = \operatorname{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ is a Dirac structure if and only if there exists a unitary matrix Θ such that $\mathcal{D} = \operatorname{ran} \left(\begin{smallmatrix} -I+\Theta \\ I+\Theta \end{smallmatrix} \right)$.

Proof: Using that $F^T E + E^T F = 0$, we find that

$$(E+F)^T(E+F) = (E-F)^T(E-F). \qquad (*)$$

So if $(E-F)v = 0$, then $(E+F)v = 0$, and thus $Ev = Fv = 0$. Since $\left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right)$ has full rank, we find $v = 0$. Thus $E - F$ is invertible. Define $\Theta = (E+F)(E-F)^{-1}$, then $(*)$ implies that Θ is unitary. Now $\mathcal{D} = \operatorname{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix} \right) = \operatorname{ran} \begin{pmatrix} 2F(E-F)^{-1} \\ 2E(E-F)^{-1} \end{pmatrix} = \operatorname{ran} \left(\begin{smallmatrix} -I+\Theta \\ I+\Theta \end{smallmatrix} \right)$. □

Dirac structures, dynamical interpretation

We assume $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$ with $\langle f \mid e \rangle = f^\top e$.

Example

If we take $F = J$, $E = I$, with $J^\top = -J$, then by the above

$$\mathcal{D} = \ker \begin{pmatrix} I^\top & J^\top \end{pmatrix} = \ker \begin{pmatrix} I & -J \end{pmatrix}$$

defines a Dirac structure.
So the solutions of $\dot{x}(t) = J\mathcal{H}x(t)$ ($\mathcal{H} = \mathcal{H}^\top$) can be seen as $\left(\begin{smallmatrix} f \\ e \end{smallmatrix} \right) = \begin{pmatrix} \dot{x}(t) \\ \mathcal{H}x(t) \end{pmatrix} \in \mathcal{D}$ and satisfy

$$\frac{d}{dt}\left[\frac{1}{2}x(t)^\top \mathcal{H}x(t)\right] = \dot{x}(t)^\top \mathcal{H}x(t) = f^\top e = 0.$$

Thus $H(t) := \frac{1}{2}x(t)^\top \mathcal{H}x(t)$ is constant along solutions of the differential equation. $\square$

Dirac structures, dynamical interpretation

Note that we did not need conditions on $\mathcal{H}$, except from symmetry.
Choosing $\mathcal{H} = \mathrm{diag}(1, -1)$ gives an unstable system, Check.
The previous example can be extended to non-linear o.d.e.'s.

Example

Let $\mathcal{D} = \mathrm{ran} \left(\begin{smallmatrix} F \\ E \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} F \\ E \end{smallmatrix}\right)$ a $2n \times n$ matrix of rank n, and with $F^\top E = -E^\top F$.

With the C^1-function $H : \mathbb{R}^n \mapsto \mathbb{R}$, we define the implicit differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \frac{\partial H}{\partial x}(x(t)) \end{pmatrix} \in \mathcal{D}.$$

Then along solutions, there holds $\frac{d}{dt} H(x(t)) = 0$. $\square$

Dirac structures, dynamical interpretation

Note that the implicit differential equation can be made explicitly as

$$E^\top \dot{x}(t) = -F^\top \frac{\partial H}{\partial x}(x(t)).$$

Since E needs not to be invertible, this includes DAE's.

The above differential equation needs not to have solutions (for all initial conditions). For instance, take $E = 0$ and $F = I$, then the diff. eqn. becomes $0 = -\frac{\partial H}{\partial x}(x(t))$.

So a Dirac structure alone does not guarantee existence nor stability.

Dirac structures, dynamical interpretation

Since there is no time in a Dirac structure, we can choose our time axis. We assume $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$ with $\langle f \mid e \rangle = f^\top e$.

Example

For $J \in \mathbb{R}^{n \times n}$ satisfying $J^\top = -J$, define the Dirac structure

$$\mathcal{D} = \ker \begin{pmatrix} I^\top & J^\top \end{pmatrix} = \ker \begin{pmatrix} I & -J \end{pmatrix}.$$

So the solutions of $x(n+1) - x(n) = J\mathcal{H}\left[x(n+1) + x(n)\right]$ ($\mathcal{H} = \mathcal{H}^\top$) can be seen as $\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) = \begin{pmatrix} x(n+1)-x(n) \\ \mathcal{H}[x(n+1)+x(n)] \end{pmatrix} \in \mathcal{D}$ and satisfy

$$\begin{aligned} x(n+1)^T \mathcal{H} x(n+1) - x(n) \mathcal{H} x(n) = \\ \left[x(n+1) - x(n)\right]^T \mathcal{H} \left[x(n+1) + x(n))\right] = 0. \end{aligned}$$

Thus $H(n) := x(n)^\top \mathcal{H} x(n)$ is constant along solutions of the difference equation. $\square$

Dirac structures, dynamical interpretation

Note that if $I - J\mathcal{H}$ is invertible, then the implicit difference equation

$$x(n+1) - x(n) = J\mathcal{H}\left[x(n+1) + x(n)\right]$$

can be made explicite. Namely, to

$$x(n+1) = (I - J\mathcal{H})^{-1}(I + J\mathcal{H})x(n).$$

Question Prove that under the conditions in the example, the matrix $I - J\mathcal{H}$ is invertible.

Dirac structures, dynamical interpretation

In the previous examples of dynamical systems we choose f to be the change of the state variable x. However, this is not dictated by the Dirac structure. Other choices are possible.

Dirac structures, dynamical interpretation

Example

We split our effort and flow space, and choose J as

$$f = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, e = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}, J = \begin{pmatrix} J_{11} & J_{12} \\ -J_{12}^\top & 0 \end{pmatrix}.$$

For $\phi_1 = \dot{x}(t)$, $\varepsilon_2 = R\phi_2$ and $\varepsilon_1 = \mathcal{H}x(t)$, $f = Je$ becomes

$$\begin{pmatrix} \dot{x}(t) \\ \phi_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ -J_{12}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H}x(t) \\ R\phi_2 \end{pmatrix}.$$

Hence x satisfies $\dot{x}(t) = (J_{11} - J_{12}RJ_{12}^\top)\mathcal{H}x(t)$. So

$$\dot{x}(t)^\top \mathcal{H}x(t) + \phi_2^\top R\phi_2 = f^\top e = 0.$$

When $R \geq 0$ this gives dissipation of $H(t) = \frac{1}{2}x(t)^\top \mathcal{H}x(t)$. $\square$

Dirac structures, dynamical interpretation

Example

We split our effort and flow space, and choose J as

$$f = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, e = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}, J = \begin{pmatrix} J_{11} & B \\ -B^\top & -J_{22} \end{pmatrix}.$$

For $\phi_1 = \dot{x}(t)$, $\phi_2 = -y(t)$, $\varepsilon_2 = u(t)$ and $\varepsilon_1 = \mathcal{H}x(t)$, $f = Je$ becomes

$$\begin{pmatrix} \dot{x}(t) \\ -y(t) \end{pmatrix} = \begin{pmatrix} J_{11} & B \\ -B^\top & -J_{22} \end{pmatrix} \begin{pmatrix} \mathcal{H}x(t) \\ u(t) \end{pmatrix}.$$

So the system

$$\dot{x}(t) = J_{11}\mathcal{H}x(t) + Bu(t)$$
$$y(t) = B^\top \mathcal{H}x(t) + J_{22}u(t)$$

satisfying $\dot{x}(t)^\top \mathcal{H}x(t) - y(t)^\top u(t) = f^\top e = 0.$ □

Dirac structures and port-Hamiltonian systems

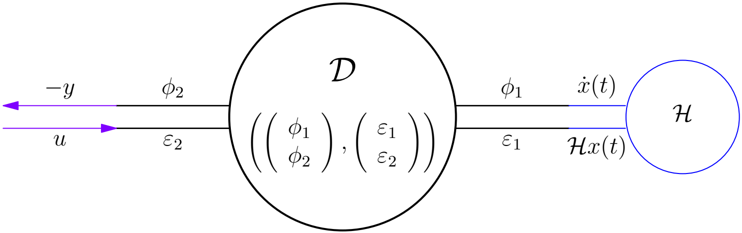

Figure: Dirac structure connected to storage, and input, output

The system

$$\dot{x}(t) = J_{11}\mathcal{H}x(t) + Bu(t)$$
$$y(t) = B^\top \mathcal{H}x(t) + J_{22}u(t)$$

is a (standard) example of a port-Hamiltonian system, with $H(t) = \frac{1}{2}x(t)^\top \mathcal{H}x(t)$ the Hamiltonian and (u, y) the ports.

Intermezzo

Intermezzo: Dirac structures and dual spaces

On our bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ we have the bilinear relation $\langle f \mid e \rangle$. If we define (for fixed f) the map

$$\ell_f : \mathcal{E} \mapsto \mathbb{R} \text{ as } \ell_f(e) = \langle f \mid e \rangle,$$

then this is a linear map from $\mathcal{E}$ to $\mathbb{R}$. Thus there exists a (unique) element $\varepsilon \in \mathcal{E}'$ (the algebraic dual of $\mathcal{E}$) such that

$$\langle f \mid e \rangle = \langle \varepsilon, e \rangle_{\mathcal{E}' \times \mathcal{E}}.$$

Hence we can define the "identification" map

$$Id : \mathcal{F} \mapsto \mathcal{E}' \text{ as } Id(f) = \varepsilon.$$

So $\mathcal{F}$ can be interpreted/identified as a subspace of $\mathcal{E}'$. Similarly, we can interpret $\mathcal{E}$ as a subspace of $\mathcal{F}'$.

Intermezzo: Dirac structures and dual spaces

When $\mathcal{F}$ and $\mathcal{E}$ are normed, linear spaces, and the bilinear product satisfies: There exists a $m > 0$ such that for all $f \in \mathcal{F}$ and $e \in \mathcal{E}$ there holds

$$|\langle f \mid e \rangle| \leq m \|f\| \|e\|.$$

Then the map

$$\ell_f : \mathcal{E} \mapsto \mathbb{R} \text{ defined as } \ell_f(e) := \langle f \mid e \rangle,$$

is a continuous/bounded linear map from $\mathcal{E}$ to $\mathbb{R}$.
Thus there exists a (unique) element $\varepsilon \in \mathcal{E}^*$ (the topological dual of $\mathcal{E}$) such that

$$\langle f \mid e \rangle = \langle \varepsilon, e \rangle_{\mathcal{E}^* \times \mathcal{E}}.$$

So $\mathcal{F}$ can be interpreted/identified as a subspace of $\mathcal{E}^*$. Similarly, we can interpret $\mathcal{E}$ as a subspace of $\mathcal{F}^*$.

End of intermezzo

Dirac structures, infinite-dimensional

We have considered $\mathcal{E}$ and $\mathcal{F}$ to be finite-dimensional, i.e., $\mathbb{R}^n$. Other (finite-dimensional) choices are possible, e.g. a tangent space and co-tangent space (see intermezzo).

Since the dimension is not "present" in the definition of a Dirac structure, we can take $\mathcal{E}$ and $\mathcal{F}$ to be infinite-dimensional.

There are many infinite-dimensional spaces, i.e., function and/or sequence spaces, and so we take a simpler approach, and try to see if we can come up with an example in which

$$V = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \mid f = Je \right\}$$

is an infinite-dimensional Dirac structure.

Dirac structures, infinite-dimensional

Classroom question:
If f and e are two (scalar) functions, what would be a logical choice for $\langle f \mid e \rangle$?
A choice is

$$\langle f \mid e \rangle = \int_\Omega f(\zeta)e(\zeta)d\zeta.$$

For simplicity, we take $\Omega = [a, b] \subseteq \mathbb{R}$.
Classroom question: Taking this bilinear product, can we think of an J such that $\{f = Je\}$ is a Dirac structure?
In particular, we need that

$$\langle f \mid e \rangle = \int_a^b f(\zeta)e(\zeta)d\zeta = \int_a^b (Je)(\zeta)e(\zeta)d\zeta = 0.$$

What about $Je = \dot{e} = \frac{de}{d\zeta}$?

Dirac structures, infinite-dimensional

We calculate

$$\begin{aligned}\langle f \mid e \rangle &= \int_a^b (Je)(\zeta)e(\zeta)d\zeta = \int_a^b \dot{e}(\zeta)e(\zeta)d\zeta \\ &= \int_a^b \frac{1}{2}\frac{d}{d\zeta}\left(e(\zeta)^2\right)d\zeta = \frac{1}{2}e(b)^2 - \frac{1}{2}e(a)^2.\end{aligned}$$

So this is only zero when we put (extra) conditions on e.
For instance, $e(b) = e(a) = 0$, or $e(b) = e(a)$, or $e(b) = -e(a)$

Dirac structures, infinite-dimensional

Question:

Given $\mathcal{F} = C(a,b)$ and $\mathcal{E} = C^1(a,b)$. Is

$$\mathcal{D}_{00} = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{F} \times \mathcal{E} \mid f = \frac{de}{d\zeta}, e(a) = 0 = e(b) \right\}$$

a Dirac structure?

Answer: Calculating $\mathcal{D}_{00}^{\perp}$;

$$\left\langle \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \mid \begin{pmatrix} f \\ e \end{pmatrix} \right\rangle = 0 \quad \forall \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D}_{00} \Leftrightarrow$$

$$\int_a^b \left[f_2(\zeta) - \dot{e}_2(\zeta) \right] e(\zeta) d\zeta = 0 \quad \forall e \in C^1(a,b).$$

Thus $f_2(\zeta) = \dot{e}_2(\zeta)$, but No boundary conditions. So $\mathcal{D}_{00}^{\perp} \neq \mathcal{D}_{00}$.

Dirac structures, infinite-dimensional

Question:
Given $\mathcal{F} = C(a, b)$ and $\mathcal{E} = C^1(a, b)$. Is

$$\mathcal{D}_p = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{F} \times \mathcal{E} \mid f = \frac{de}{d\zeta}, e(a) = e(b) \right\}$$

a Dirac structure?
Answer: Calculating $\mathcal{D}_p^\perp$ leads to (see previous example);
$\forall e \in C^1(a, b)$:

$$\begin{aligned} 0 &= \int_a^b \left[f_2(\zeta) - \dot{e}_2(\zeta)\right] e(\zeta) d\zeta + e_2(b)e(b) - e_2(a)e(a) \\ &= \int_a^b \left[f_2(\zeta) - \dot{e}_2(\zeta)\right] e(\zeta) d\zeta + [e_2(b) - e_2(a)]e(a). \end{aligned}$$

Thus $f_2(\zeta) = \dot{e}_2(\zeta)$ and $e_2(b) = e_2(a)$. So $\mathcal{D}_p^\perp = \mathcal{D}_p$.

Dirac structures, link to pde's

So $\mathcal{D}_p = \left\{ \left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{F} \times \mathcal{E} \mid f = \frac{de}{d\zeta}, e(a) = e(b) \right\}$ is a Dirac structure. As we did in the finite-dimensional case we can link a differential equation to it, by choosing $f = \dot{x}(t)$ and $e = \mathcal{H}x(t)$. Since e and f depend on ζ and thus $x(t)$ depends on ζ as well, we now have to write $f = \frac{\partial x}{\partial t}$.
$(f, e) \in \mathcal{D}_p$ is now the same as writing $\mathcal{H}(\cdot)x(\cdot, t) \in C^1(a, b)$ and

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta} \left[\mathcal{H}(\zeta) x(\zeta, t) \right], \quad \mathcal{H}(a)x(a, t) = \mathcal{H}(b)x(b, t).$$

So a PDE with Boundary Conditions.
The Dirac structure gives (as before) that along solutions we have $H(t) = \frac{1}{2} \int_a^b x(\zeta, t) \mathcal{H}(\zeta) x(\zeta, t) d\zeta$ is constant.

Dirac structures, link to pde's

Again we have seen that the Dirac structure implies properties of the solution of a differential equation, but do we have solutions?
We take $\mathcal{H} = 1$. So our scalar PDE becomes

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \quad x(a, t) = x(b, t).$$

The solution of this PDE is

$$x(\zeta, t) = x_{0,ext}(t + \zeta)$$

with $x_{0,ext}$ the periodic extension of x_0 (the initial condition).
Even when $x_0 \in \mathcal{E} = C^1(a, b)$, $x(t, \cdot) \notin \mathcal{E}$!
Once more we see that a Dirac structure does not guarantee existence of solutions. More later.

Dirac structures, P_1 class

We have studied Dirac structures for scalar functions, and we can easily extend it to vector valued functions.
So the bilinear product becomes for $f(\zeta), e(\zeta) \in \mathbb{R}^n$, $\zeta \in [a, b]$

$$\langle f \mid e \rangle = \int_a^b f(\zeta)^\top e(\zeta) d\zeta.$$

Question
Simplify $\langle f \mid e \rangle$ when $f = P_1 \frac{de}{d\zeta}$ with $P_1^\top = P_1 \in \mathbb{R}^{n \times n}$.
Answer

$$\int_a^b \left[P_1 \frac{de}{d\zeta}(\zeta) \right]^\top e(\zeta) d\zeta = \frac{1}{2} \left[e(b)^\top P_1 e(b) - e(a)^\top P_1 e(a) \right].$$

Dirac structures, P_1 class

So to make $V = \{f = P_1 \frac{de}{d\zeta}\}$ into a Dirac structure, we have to add boundary conditions.
Therefor we define boundary flow and effort

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{R_0} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix}.$$

Question: Show that for $f = P_1 \frac{de}{d\zeta} + P_0 e$, with $P_k \in \mathbb{R}^{n \times n}$, $P_1^\top = P_1$, $P_0^\top = -P_0$, there holds

$$\langle f \mid e \rangle - f_\partial^\top e_\partial = 0.$$

Question: For $\mathcal{F} = C([a,b]; \mathbb{R}^n)$ and $\mathcal{E} = C^1([a,b]; \mathbb{R}^n)$ define a Dirac structure around $f = P_1 \frac{de}{d\zeta} + P_0 e$. Furthermore, prove that your candidate Dirac structure is a Dirac structure.

Dirac structures, P_1, P_0 class

Question: For $\mathcal{F} = C([a,b];\mathbb{R}^n) \times \mathbb{R}^n$ and $\mathcal{E} = C^1([a,b];\mathbb{R}^n) \times \mathbb{R}^n$ define a Dirac structure around $f = P_1 \frac{de}{d\zeta} + P_0 e$. Furthermore, prove that your candidate Dirac structure is a Dirac structure.

The PDE associated to the above Dirac structure will be $\mathcal{H}(\cdot)x(\cdot,t) \in C^1([a,b];\mathbb{R}^n)$ and

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left[\mathcal{H}(\zeta)x(\zeta,t)\right] + P_0 \mathcal{H}(\zeta)x(\zeta,t)$$

with (in)homogeneous boundary conditions.
The existence problem which we found in the scalar case remains.

Dirac structures, P_1, P_0 class

As an example of the P_1 class we take $n = 2$, $P_0 = 0$,

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H}(\zeta) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Thus

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\partial x}{\partial t} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta} \left[\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} x \right] = \frac{\partial}{\partial \zeta} \begin{pmatrix} c x_2 \\ c x_1 \end{pmatrix}.$$

For x_1 this becomes

$$\frac{\partial^2 x_1}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\partial x_1}{\partial t} \right] = \frac{\partial}{\partial t} \left[\frac{\partial c x_2}{\partial \zeta} \right] = c \frac{\partial}{\partial \zeta} \left[\frac{\partial x_2}{\partial t} \right] = c^2 \frac{\partial^2 x_1}{\partial \zeta^2}$$

The wave equation.

Dirac structures, higher spatial dimension.

We have only discussed (potential) infinite-dimensional Dirac structures in one spatial variable. However, there is no fundamental reason for that. For instance, consider

$$V = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right\}$$

with $\Omega \subset \mathbb{R}^3$, and $e_1 \in C^1(\Omega; \mathbb{R})$, $e_2 \in C^1(\Omega; \mathbb{R}^3)$, etc.

$$\begin{aligned} \langle f \mid e \rangle &= \int_\Omega e_1 \mathrm{div}(e_2) + e_2^\top \mathrm{grad}(e_1) \\ &= \int_\Omega \mathrm{div}(e_1 e_2) = \int_\Gamma (e_1 e_2)^\top n, \end{aligned}$$

where Γ is the boundary of Ω and n is the outward unit normal.

Dirac structures, so far.

We have seen that using functions paces, we can define Dirac structures. Furthermore, we can link these infinite-dimensional Dirac structures to (partial) differential equations. However, we have trouble (even in simple cases) to obtain existence of solutions for these PDE's.
To solve this matter we take a more abstract/functional analytic point of view.

Dirac structure, operators

For finite-dimensional spaces we had that $\{f = Je\}$ defines a Dirac structure if and only if $J^\top = -J$. How for infinite-dimensional spaces?

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $Q : \mathrm{dom}(Q) \subseteq X \mapsto X$ be a densely defined linear operator.

Definition

The adjoint, Q^*, of Q is defined as follows

$$\mathrm{dom}(Q^*) = \{z \in X \mid \exists w \in X \text{ s.t. } \langle Qx, z \rangle = \langle x, w \rangle, \forall x \in \mathrm{dom}(Q)\}$$

For $z \in \mathrm{dom}(Q^*)$, we define $Q^*(z) = w$. □

Definition

- Q is skew-adjoint when $Q^* = -Q$.
- Q is self-adjoint when $Q^* = Q$.

Dirac structure, operators

Theorem

Let $\mathcal{F} = \mathcal{E} = X$, with X a Hilbert space, and let $\langle f \mid e \rangle = \langle f, e \rangle_X$. Then

$$\mathcal{D} = \{ \left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{F} \times \mathcal{E} \mid f = Je, e \in \mathrm{dom}(J) \}$$

is a Dirac structure if and only if J is skew-adjoint.

Proof: Calculating $\mathcal{D}^\perp$;

$$\left\langle \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \mid \begin{pmatrix} f \\ e \end{pmatrix} \right\rangle = 0 \quad \forall \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D} \Leftrightarrow$$

$$\langle f_2, e \rangle_X + \langle Je, e_2 \rangle_X = 0 \quad \forall e \in \mathrm{dom}(J).$$

So $e_2 \in \mathrm{dom}(J^*)$ and $f_2 = -J^*(e_2)$. □

Dirac structure, revisited

We have seen that

$$\mathcal{D}_p = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in C(a,b) \times C^1(a,b) \mid f = \frac{de}{d\zeta}, e(a) = e(b) \right\}$$

is a Dirac structure. However,

- $\mathcal{F} \neq \mathcal{E}$;
- $\mathcal{F}$ nor $\mathcal{E}$ is a Hilbert space,

but $f = \frac{de}{d\zeta}$ looks very similar to $f = Je$. Furthermore, the bilinear product $\int_a^b f(\zeta)e(\zeta)d\zeta$ looks very similar to an inner product. Namely, the inner product of $L^2(a,b)$-functions.

Dirac structure, revisited

So we take

- $\mathcal{F} = \mathcal{E} = L^2(a,b)$ all measurable, square integrable, real-valued, scalar functions on the interval (a,b);
- $Je = \frac{de}{d\zeta}$ with $\mathrm{dom}(J) = \{e \in H^1(a,b) \mid e(a) = e(b)\}$.

Then J is skew-adjoint, and thus

$$\mathcal{D}_{per} = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in L^2(a,b) \times H^1(a,b) \mid f = \frac{de}{d\zeta}, e(a) = e(b) \right\}$$

is a Dirac structure.

Dirac structures, from infinite- to finite-dimensional

Given an infinite-dimensional Dirac structure of the form

$$\mathcal{D}_\infty = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{F} \times \mathcal{E} \mid f = Je \right\}$$

we can easily obtain a finite-dimensional Dirac structure. Therefor we choose $e_1, \cdots, e_N$ (independent) elements of $\mathcal{E}$, and define $f_k = Je_k$, $k = 1, \cdots N$. Next define

- $\mathcal{E}_N := \mathrm{span}\{e_1, \cdots, e_N\} \subset \mathcal{E}$;
- $\mathcal{F}_N := \mathrm{span}\{f_1, \cdots, f_N\} \subset \mathcal{F}$;
- For $(f, e) \in \mathcal{F}_N \times \mathcal{E}_N$ the bilinear product is defined as $\langle f \mid e \rangle_N := \langle f \mid e \rangle$.
- $\mathcal{D}_N := \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{F}_N \times \mathcal{E}_N \mid f = Je \right\} \subset \mathcal{D}$

Question: Prove that $\mathcal{D}_N$ is a Dirac structure in $\mathcal{F}_N \times \mathcal{E}_N$ if and only if $\dim \mathcal{F}_N = N$.

Dirac structures, from infinite- to finite-dimensional

As an example we consider

$$\mathcal{D}_{per} = \left\{ \begin{pmatrix} f \\ e \end{pmatrix} \in L^2(0,1) \times H^1(0,1) \mid f = \frac{de}{d\zeta}, e(0) = e(1) \right\}.$$

We choose $N \in \mathbb{N}$ and define $h = N^{-1}$. Furthermore $\zeta_k := k * h$, $k = 0, 1, \cdots, N$. With this we define "hat" functions

$$e_k(\zeta) = \begin{cases} N(\zeta - \zeta_{k-1}) & \zeta \in [\zeta_{k-1}, \zeta_k]; \\ N(\zeta_{k+1} - \zeta) & \zeta \in [\zeta_k, \zeta_{k+1}]; \\ 0 & \text{elsewhere.} \end{cases}$$

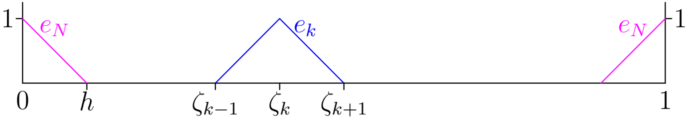

Figure: The hat-functions, e_k

Dirac structures, from infinite- to finite-dimensional

From $f_k = Je_k = \frac{de_k}{d\zeta}$, we find

$$f_k(\zeta) = \begin{cases} N & \zeta \in (\zeta_{k-1}, \zeta_k); \\ -N & \zeta \in (\zeta_k, \zeta_{k+1}); \\ 0 & \text{elsewhere.} \end{cases}$$

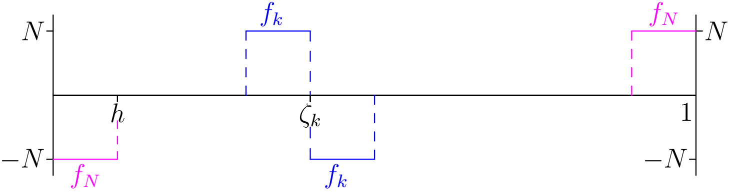

Figure: The step-functions, f_k

It is easy to show that $\dim\left(\mathrm{span}_{k=1,\cdots,N}\{f_k\}\right) = N$, and thus $\mathcal{D}_N = \{\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{F}_N \times \mathcal{E}_N \mid f = Je\}$ is a Dirac structure (finite-dimensional).

Dirac structures, from infinite- to finite-dimensional

Since $\dim \mathcal{E}_N = \dim \mathcal{F}_N = N$, we can define an equivalent Dirac structure on $\mathbb{R}^N \times \mathbb{R}^N$.
For $e \in \mathcal{E}_N$ and $f \in \mathcal{F}_N$ given as $e(\zeta) = \sum_{k=1}^{N} a_k e_k(\zeta)$ and $f(\zeta) = \sum_{k=1}^{N} b_k f_k(\zeta)$, respectively, we define

$$\vec{\mathbf{e}} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \quad \vec{\mathbf{f}} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}.$$

By definition, $f_k = Je_k$. Thus the Dirac structure, becomes

$$\mathcal{D}_N = \left\{ \begin{pmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{e}} \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N \mid \vec{\mathbf{f}} = \vec{\mathbf{e}} \right\}.$$

Question: Something weird and/or wrong?

Dirac structures, from infinite- to finite-dimensional

A straightforward calculation gives

$$\langle f_k \mid e_\ell \rangle = \begin{cases} -\frac{1}{2}N^2 & k = \ell + 1 \\ \frac{1}{2}N^2 & k = \ell - 1 \\ 0 & \text{elsewhere} \end{cases}$$

So

$$\langle f \mid e \rangle \neq \vec{\mathbf{f}}^\top \vec{\mathbf{e}},$$

but

$$\langle f \mid e \rangle = \vec{\mathbf{f}}^\top Q \vec{\mathbf{e}}$$

with $Q_{k,l} = \langle f_k \mid e_\ell \rangle$. (see also Question on Page 7)

Port Hamiltonian systems from analysis to numerics

Abstract Differential Equations

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Introduction and notation

In this part we go into existence theory for linear PDE's. We will focus on those on a one-dimensional spatial domain, and will study homogeneous and inhomogeneous equations.

Some notation:

- In this part we denote the one dimensional spatial domain by $[0, \ell]$. Hence we shifted it by a. However, we have kept units (which can be lost when choosing the interval $[0, 1]$).
- The norm on the inner Hilbert space X we denote by $\| \cdot \|$ and the inner product by $\langle \cdot, \cdot \rangle$

Solutions of PDE's

To introduce and motivate solutions of a PDE, we consider the following simple PDE with $\zeta \in [0, \ell]$ and $t \geq 0$

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial w}{\partial \zeta}(\zeta, t), \quad w(\ell, t) = 0, \quad w(\zeta, 0) = w_0(\zeta).$$

We call a function $w : [0, \ell] \times [0, \infty) \to \mathbb{R}$ a classical solution, if w is continuously differentiable, and for all $t \geq 0$, $\zeta \in [0, \ell]$ the differential equation, initial and boundary condition are satisfied.

Question: Determine the classical solution for $w_0(\zeta) = \sin(\pi\zeta/\ell)$.

History has shown that this concept is too restrictive, and that a weaker concept of a solution was needed. We illustrate this for the same PDE.

Solutions of PDE's

We take a smooth test function $\phi(\zeta)$ and integrate over the spatial domain.

$$\begin{aligned}
\int_0^\ell \phi(\zeta)\frac{\partial w}{\partial t}(\zeta,t)d\zeta &= \int_0^\ell \phi(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)d\zeta \quad \text{(PDE)} \\
\text{(int. by parts)} \quad &= [\phi(\zeta)w(\zeta,t)]_0^\ell - \int_0^\ell \dot{\phi}(\zeta)w(\zeta,t)d\zeta \\
\text{(b.c.)} \quad &= -\phi(0)w(0,t) - \int_0^\ell \dot{\phi}(\zeta)w(\zeta,t)d\zeta.
\end{aligned}$$

If we take test functions satisfying $\phi(0) = 0$, we find

Solutions of PDE's

$$\frac{d}{dt}\int_0^\ell \phi(\zeta)w(\zeta,t)d\zeta = \int_0^\ell \phi(\zeta)\frac{\partial w}{\partial t}(\zeta,t)d\zeta = -\int_0^\ell \dot{\phi}(\zeta)w(\zeta,t)d\zeta.$$

Integrate this expression with respect to time from $t = 0$ to $t = t_f$

$$\int_0^\ell \phi(\zeta)w(\zeta,t_f)d\zeta - \int_0^\ell \phi(\zeta)w(\zeta,0)d\zeta = -\int_0^{t_f}\int_0^\ell \dot{\phi}(\zeta)w(\zeta,t)d\zeta.$$

You see there are no derivatives of w taken anymore.
Now we call $w(\zeta,t)$ a weak or mild solution of the PDE if the above equation is satisfied for all smooth test functions ϕ satisfying $\phi(0) = 0$.
The set of initial conditions must be chosen. With this you also choose the set in which $w(\cdot,t_f)$ will be. We denote this (linear) space by X.

Weak and classical solutions of PDE's

Question For a given $w_0 \in X = L^2(0, \ell)$ show that

$$w(\zeta, t) = \begin{cases} w_0(\zeta + t) & \zeta + t \in [0, \ell] \\ 0 & \text{elsewhere} \end{cases}$$

is the weak solution of

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial w}{\partial \zeta}(\zeta, t), \quad w(\ell, t) = 0, \quad w(\zeta, 0) = w_0(\zeta).$$

Weak and classical solutions of PDE's

It is easy to see that a classical solution is always a weak solution, but the converse need not to hold.
We will now study when our PDE has a weak solution.
Note there is a difference between knowing the existence of a solution and having the form/expression of the solution. The expression for the solution can be hard/impossible to find. So we concentrate on existence.
We concentrate on solutions satisfying the additional property that

$$\|x(t)\| \leq \|x_0\| \quad \forall t > 0 \qquad \text{(contraction)},$$

where $\|\cdot\|$ denotes the norm of the state space X.
Since our PDE's are linear, the above inequality implies that the solution with depend continuously on the initial condition, i.e., for all $t \geq 0$

$$\|x_1(t) - x_2(t)\| \leq \|x_{10} - x_{20}\| \quad \text{(continuity w.r.t. initial condition)}.$$

Intermezzo

Intermezzo: Strongly continuous semigroups

Consider a linear, time invariant differential equation on the space X. Assume that for every $x_0 \in X$ there exists a (weak) solution denoted by $x(t)$. Furthermore, assume that this solution depends continuously on the initial condition.
Define for $t \geq 0$ the map $T(t) : X \mapsto X$ as

$$T(t)x_0 = x(t).$$

Then it has the following properties:

- $T(0) = I$;
- $T(t_1 + t_2) = T(t_1)T(t_2)$, $t_1, t_2, \in [0, \infty)$, time-invariance;
- $T(t)$ is for every $t \geq 0$ a linear and bounded operator, i.e., $T(t) \in \mathcal{L}(X)$.

If additionally the following holds

$$\lim_{t \downarrow 0} \|T(t)x_0 - x_0\| = 0, \quad \text{continuity at } t = 0,$$

then $(T(t))_{t \geq 0}$ is a strongly continuous semigroup, or short C_0-semigroup.

Intermezzo: Strongly continuous semigroups, examples

It is not hard to show that on $X = \mathbb{R}^n$ the exponential e^{At} is a C_0-semigroup.
Question Show that the solution map of the PDE

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial w}{\partial \zeta}(\zeta, t), \quad w(\ell, t) = 0, \quad w(\zeta, 0) = w_0(\zeta).$$

is a C_0-semigroup.

Intermezzo: Strongly continuous semigroups

Since $T(t)$ came from $x(t)$ via $x(t) = T(t)x_0$, we have

$$\dot{x}(t) = \lim_{h \downarrow 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \downarrow 0} \frac{T(t+h)x_0 - T(t)x_0}{h}.$$

Thus by the semigroup and boundedness property,

$$\dot{x}(t) = \lim_{h \downarrow 0} \frac{T(t)T(h)x_0 - T(t)x_0}{h} = T(t) \lim_{h \downarrow 0} \frac{T(h)x_0 - x_0}{h}.$$

We define (whenever it exists)

$$Ax_0 := \lim_{h \downarrow 0} \frac{T(h)x_0 - x_0}{h}.$$

With this we obtain the (abstract) differential equation

$$\dot{x}(t) = T(t)Ax_0 = AT(t)x_0 = Ax(t).$$

Intermezzo: Abstract differential equation

For the linear operator A we denote its domain by dom(A).
Given now a linear operator $A : \mathrm{dom}(A) \subseteq X \mapsto X$, under which conditions does the abstract differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

have solution, i.e., when do we have the existence of a C_0-semigroup?
For X being a Hilbert space (from now on standard assumption) we have the following:

Intermezzo: Abstract differential equation

Theorem

If A is skew-adjoint, i.e., $A^* = -A$, then A generates a C_0-semigroup satisfying

- $\|T(t)\| = 1$ *for all* $t \geq 0$;
- $T(t)$ *can be extended to the whole real time, and* $T(t_1 + t_2) = T(t_1)T(t_2)$, $t_1, t_2 \in \mathbb{R}$ *and* $\|T(t)\| = 1$ *for all* $t \in \mathbb{R}$, *unitary group*.

Theorem

If A is dissipative, i.e., $\langle Ax, x\rangle \leq 0 \; \forall x \in \mathrm{dom}(A)$, and if A^* is dissipative, then A generates a C_0-semigroup satisfying $\|T(t)\| \leq 1$ for all $t \geq 0$; contraction semigroup.

For $x_0 \in \mathrm{dom}(A)$ the function $x(t) = T(t)x_0$ a classical solution.
For $x_0 \in X$ it is a weak solution.

Intermezzo: Useful lemma

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $Q \in \mathcal{L}(X)$ satisfying $Q = Q^*$, and $\langle x, Qx \rangle \geq m\|x\|^2$, $\forall x \in X$.

Question: Prove that if J is skew-adjoint in X, then JQ is skew-adjoint in the inner product $\langle x, z \rangle_Q := \langle x, Qz \rangle$.

End of intermezzo

Introduction

We have now the right basis in operator theory/functional analysis and PDE theory to study the existence of solutions for a PDE with an underlying Dirac structure. We had:

Theorem

Let $\mathcal{F} = \mathcal{E} = X$, with X a Hilbert space, and let $\langle f \mid e \rangle = \langle f, e \rangle_X$. Then

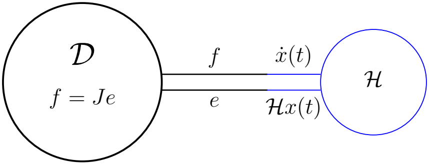
$$\mathcal{D} = \{ \left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \mathcal{F} \times \mathcal{E} \mid f = Je, e \in \mathrm{dom}(J) \}$$

is a Dirac structure if and only if J is skew-adjoint.

Furthermore: a skew-adjoint J generates a C_0-semigroup (unitary group) on the Hilbert space X.

Dirac and ADE

Let J be skew-adjoint on the Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ and consider the abstract differential equation, given as



Question: Does the corresponding abstract differential equation

$$\dot{x}(t) = J\mathcal{H}x(t), \qquad x(0) = x_0$$

possess a (unique) solution for all $x_0 \in X$?
Yes, but we need that $mI \leq \mathcal{H} \leq MI$ for some $m, M > 0$.
If $\frac{1}{2}\langle x, \mathcal{H}x \rangle$ has the meaning "energy", then the solution exists for every initial condition with finite energy, and the energy stays constant along the solution.

Dirac and PDE

For our class of PDE's on the spatial interval $[0, \ell]$

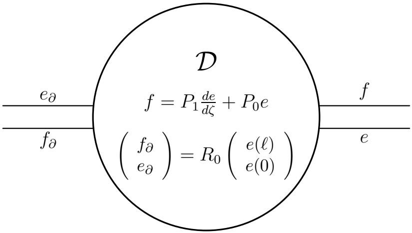
$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left[\mathcal{H}(\zeta) x(\zeta, t)\right] + P_0 \mathcal{H}(\zeta) x(\zeta, t),$$

we have the associated Dirac structure

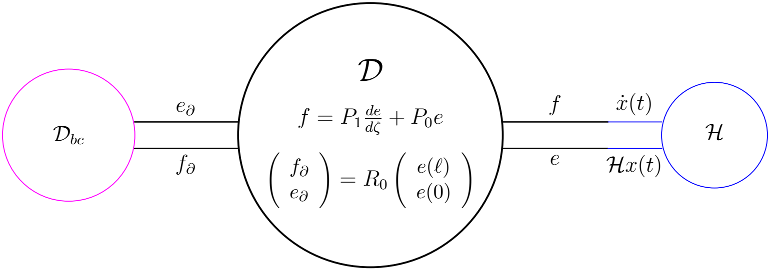
$$\mathcal{D} = \left\{ f = P_1 \frac{de}{d\zeta} + P_0 e, \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right) = \underbrace{\frac{1}{\sqrt{2}} \left(\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right)}_{R_0} \left(\begin{smallmatrix} e(b) \\ e(a) \end{smallmatrix}\right) \right\}.$$

We take $\mathcal{F} = \mathcal{E} = L^2(0, \ell)$, $\langle f | e \rangle = \langle f, e \rangle$, and in $\mathcal{D}$ we restrict e to $H^1(0, \ell)$.

Dirac and PDE

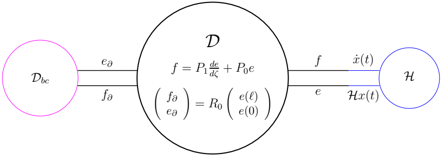


We connect it at one end to a Hamiltonian, and on the other end to another Dirac structure.



Dirac and PDE

The PDE associated to the connection



is given as

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left[\mathcal{H}(\zeta) x(\zeta, t)\right] + P_0 \mathcal{H}(\zeta) x(\zeta, t),$$

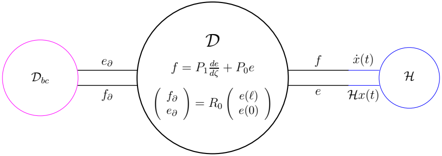
with boundary condition

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \operatorname{ran} \begin{pmatrix} F \\ E \end{pmatrix}.$$

Dirac and PDE

Theorem (Le Gorrec, Maschke & Z. '05)

Assume that $P_0 = -P_0^\top$, $P_1 = P_1^\top$, P_1 *invertible and* $0 < mI \leq \mathcal{H}(\zeta) \leq MI$, *for all* $\zeta \in [0, \ell]$. *Then the PDE associated to the connection*



has for every $x_0 \in X$ *a unique weak solution satisfying* $\|x(t)\|_{\mathcal{H}} = \|x_0\|_{\mathcal{H}}$, $t \in \mathbb{R}$,
Or equivalently, the associated A *generates a unitary group on* $L^2([0, \ell]; \mathbb{R}^n)$ *with energy norm* $\|x\|^2_{\mathcal{H}} = \langle x, \mathcal{H}x \rangle$.

Dirac and PDE

Question: How many boundary conditions does the previous PDE have?
Question: Define the Hamiltonian $H(t) := \frac{1}{2}\langle x(t), \mathcal{H}x(t)\rangle$. What do you know about $\dot{H}(t)$?

Note that the boundary conditions can be written in the more familiar form

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \ker \begin{pmatrix} E^T & F^T \end{pmatrix},$$

or

$$\begin{pmatrix} I + \Theta & I - \Theta \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0,$$

with Θ unitary.
With this, the previous theorem can be reformulated.

Solution to pH-PDE

Theorem (Le Gorrec, Maschke & Z. '05, Jacob & Z '11)

Given our port-Hamiltonian partial differential equation

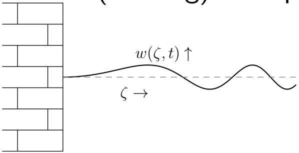
$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H}(\zeta) x(\zeta, t)]$$

with the properties on P_0, P_1 *and* $\mathcal{H}$, *and boundary conditions* $W_B \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right) = 0$, W_B *a* $n \times 2n$*-matrix. Then the following are equivalent:*

- *The PDE has for every* $x_0 \in X$ *a unique weak solution satisfying* $\|x(t)\|_{\mathcal{H}} = \|x_0\|_{\mathcal{H}}$, $t \in \mathbb{R}$;
- W_B *can be written as* $S \begin{pmatrix} I + \Theta & I - \Theta \end{pmatrix}$ *with* S *invertible and* Θ *unitary;*
- W_B *has full rank, and* $\dot{H}(0) = 0$ *for all (smooth) initial conditions satisfying the boundary conditions.*

Example

As our (running) example we consider the vibrating string



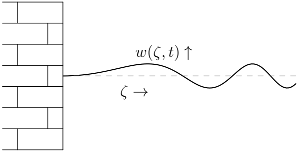
$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right].$$

With ρ the mass density, and T Young's modulus.
We choose $x_1 := \rho\frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain), and write the PDE as

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(\zeta,t) = \underbrace{\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=P_1}\frac{\partial}{\partial \zeta}}_{J}\left[\underbrace{\begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}}_{=\mathcal{H}} x(\zeta,t)\right].$$

Boundary conditions and power balance

Our vibrating string



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right].$$

is fixed at $\zeta = 0$ and moves freely at $\zeta = \ell$. In the state variables $x_1 = \rho\frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$ this gives the (boundary) conditions

$$x_1(0,t) = 0 \text{ and } x_2(\ell,t) = 0.$$

The power balance becomes

$$\begin{aligned}\dot{H}(t) =& \frac{1}{2}\left[(\mathcal{H}x)^T(\zeta,t)P_1(\mathcal{H}x)(\zeta,t)\right]_0^\ell \\ =& \frac{1}{2}\left[\begin{pmatrix}\frac{1}{\rho(\zeta)}x_1(\zeta,t)\\ T(\zeta)x_2(\zeta,t)\end{pmatrix}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\rho(\zeta)}x_1(\zeta,t)\\ T(\zeta)x_2(\zeta,t)\end{pmatrix}\right]_0^\ell = 0.\end{aligned}$$

Example: the wave equation

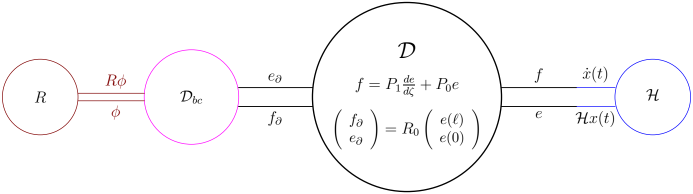
Now we check the conditions.

- $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an invertible 2×2 matrix ($n = 2$).
- $P_0 = 0$, so skew-symmetric.
- If $0 < m \leq T(\zeta), \rho(\zeta)^{-1} \leq M$ for all ζ, then $\mathcal{H}(\zeta) = \begin{pmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & T(\zeta) \end{pmatrix}$ satisfies $mI_2 \leq \mathcal{H}(\zeta) \leq MI_2$.
- $W_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} R_0^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$ has rank 2.
- $\dot{H}(0) = 0$.

Thus our pH system has for every $x_0 \in X$ a unique weak solution for $t \in \mathbb{R}$ with constant energy.

Dirac and PDE

Assume that we add a damping to the left hand side of $\mathcal{D}_{bc}$.



Question: What would now hold for $\dot{H}(t)$?

Solution to pH-PDE

Theorem (Le Gorrec, Maschke & Z. '05, Jacob & Z '11)

Given our port-Hamiltonian partial differential equation

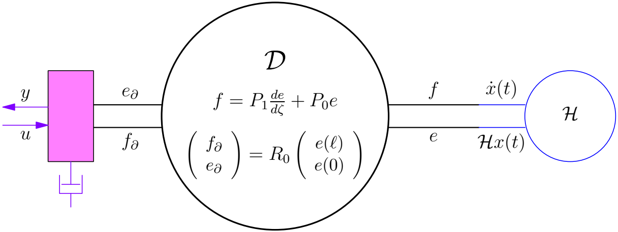
$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)]$$

with the properties on P_0, P_1 *and* $\mathcal{H}$, *and boundary conditions* $W_B \left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right) = 0$, W_B *a* $n \times 2n$*-matrix. Then the following are equivalent:*

- *The PDE has for every* $x_0 \in X$ *a unique weak solution satisfying* $\|x(t)\|_{\mathcal{H}} \leq \|x_0\|_{\mathcal{H}}$, $t \geq 0$, *i.e, a contraction semigroup;*
- W_B *can be written as* $S\begin{pmatrix} I+V & I-V \end{pmatrix}$ *with* S *invertible and* V *satisfies* $VV^\top \leq I$*;*
- W_B *has full rank, and* $\dot{H}(0) \leq 0$ *for all (smooth) initial conditions satisfying the boundary conditions.*

Input and outputs

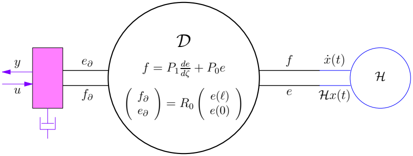
We don't only want to study homogeneous PDE's, but also want to allow for control/inputs and observations/outputs. Assume that we add an input and output to the left hand side of $\mathcal{D}_{bc}$.



This is a port-Hamiltonian system with damping, and inputs/outputs.

Input and outputs

The partial differential equation associated to



is

$$\frac{\partial x}{\partial t}(\zeta, t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H}(\zeta) x(\zeta, t)];$$

$$0 = W_{B,1} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix};$$

$$u(t) = W_{B,2} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix};$$

$$y(t) = W_C \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}.$$

Solution to inhomogeneous pH-PDE

Theorem (Z, Le Gorrec, Maschke & Villegas '10, Jacob & Z '11)

Given our port-Hamiltonian partial differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)\left[\mathcal{H}(\zeta)x(\zeta,t)\right]$$

with the properties on P_0, P_1 *and* $\mathcal{H}$, *and boundary conditions, input and outputs*

$$\begin{pmatrix} W_{B,1} \\ W_{B,2} \\ W_C \end{pmatrix} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \begin{pmatrix} 0 \\ u(t) \\ y(t) \end{pmatrix}$$

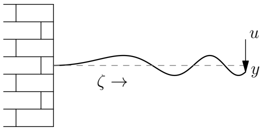
with $W_B := \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}$ *a full rank* $n \times 2n$*-matrix. If there exists a unique weak solution when* $u \equiv 0$, *then for every initial condition in* X *and every* $u \in L^2((0,t_1);\mathbb{R}^m)$ *there is a unique solution with* $y \in L^2((0,t_1);\mathbb{R}^k)$, $t_1 > 0$ *arbitrary.*

Solution to inhomogeneous pH-PDE

Comments

- Note that we have simple condition for existence of the homogeneous PDE.
- It is standard "PDE-theory" to show that for sufficiently smooth inputs you have existence, see [Le Gorrec, Maschke & Z '05].
- The proof of this theorem is based on a result by G. Weiss from 1994.

Example: the wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$\frac{\partial w}{\partial t}(0,t) = 0, \quad T(\ell)\frac{\partial w}{\partial \zeta}(\ell,t) = u(t)$$

$$\frac{\partial w}{\partial t}(\ell,t) = y(t).$$

So we control the force and measure the velocity at the right end. Since have shown that for $u \equiv 0$ we have a solution (even a unitary group), we have a unique (weak) solution for all initial conditions in X and every $u \in L^2(0,t_1)$.

Transfer function, general

Let Σ be a system with input $u(t)$, output $y(t)$ and remaining variables $z(t)$.
Let $s \in \mathbb{C}$ and $u_0 \in U$ (input (value) space) be given. If there exists a solution $(u(t), z(t), y(t)$ of the form $(u(t), z(t), y(t)) = (u_0 e^{st}, z_0 e^{st}, y_0 e^{st})$, then this is called an exponential solution.
Let $s \in \mathbb{C}$ be given. If for every $u_0 \in U$, there exists a (unique) exponential solution, then the map $G(s) : U \mapsto Y$, $G(s)u_0 = y_0$ is called the transfer function at s of the system Σ.

Transfer function, pH-PDE

For our pH-PDE

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1\frac{\partial}{\partial\zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)];$$

$$0 = W_{B,1}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix}; \qquad u(t) = W_{B,2}\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix};$$

$$y(t) = W_C\begin{pmatrix} f_\partial(t) \\ e_\partial(t)\end{pmatrix}.$$

the transfer function is found by solving for given $u_0 \in \mathbb{R}^m$, $s \in \mathbb{C}$

$$\frac{\partial x_0(\zeta)e^{st}}{\partial t} = \left(P_1\frac{\partial}{\partial\zeta} + P_0\right)\left[\mathcal{H}(\zeta)x_0(\zeta)e^{st}\right];$$

$$0 = W_{B,1}\begin{pmatrix} f_{\partial,0}e^{st} \\ e_{\partial,0}e^{st}\end{pmatrix}; \qquad u_0e^{st} = W_{B,2}\begin{pmatrix} f_{\partial,0}e^{st} \\ e_{\partial,0}e^{st}\end{pmatrix};$$

$$y_0e^{st} = W_C\begin{pmatrix} f_{\partial,0}e^{st} \\ e_{\partial,0}e^{st}\end{pmatrix}.$$

Transfer function, pH-PDE

This is the same as solving

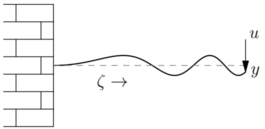
$$sx_0(\zeta) = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}(\zeta)x_0(\zeta)];$$

$$0 = W_{B,1}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right); \qquad u_0 = W_{B,2}\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right);$$

$$y_0 = W_C\left(\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0} \end{smallmatrix}\right).$$

This is almost always impossible. However, the balance equation can give properties of the transfer function $G(s)$.

Example: the transfer function of the wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$\frac{\partial w}{\partial t}(0,t) = 0, \quad T(\ell)\frac{\partial w}{\partial \zeta}(\ell,t) = u(t)$$

$$\frac{\partial w}{\partial t}(\ell,t) = y(t).$$

We have, see before,

$$\begin{aligned}\dot{H}(t) &= f_\partial(t)e_\partial(t)\\ &= T(\ell)\frac{\partial w}{\partial \zeta}(\ell,t)\frac{\partial w}{\partial t}(\ell,t) - T(0)\frac{\partial w}{\partial \zeta}(0,t)\frac{\partial w}{\partial t}(0,t)\\ &= u(t)y(t).\end{aligned}$$

Since $H(t) = \langle x(t), \mathcal{H}x(t)\rangle$, we find

Example: the transfer function of the wave equation

For every solution of this controlled and observed vibrating string

$$\dot{H}(t) = \frac{d}{dt}\langle x(t), \mathcal{H}x(t)\rangle = u(t)y(t).$$

Substituting the exponential solution, we have

$$\dot{H}(t) = \frac{d}{dt}\langle x_0e^{st}, \mathcal{H}x_0e^{st}\rangle = u_0e^{st}y_0e^{st}.$$

$$2s\langle x_0e^{st}, \mathcal{H}x_0e^{st}\rangle = u_0e^{st}y_0e^{st} \quad \Leftrightarrow$$
$$2s\langle x_0, \mathcal{H}x_0\rangle = u_0y_0 = u_0G(s)u_0 = G(s)u_0^2.$$

Since $\langle x_0, \mathcal{H}x_0\rangle \geq 0$, we find $G(s) > 0$ for $s > 0$.
G is "positive real".

C'est tout